Example: 2D Quadratic Drag

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$$\vec{F}_D = -D\vec{v}|\vec{v}|$$

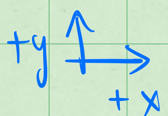
Need coordinate system.

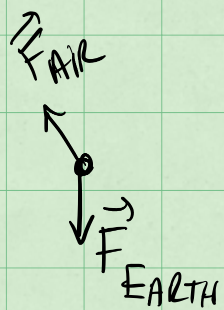
$$\vec{r} = x\hat{x} + y\hat{y} = \langle x, y\rangle$$

$$\vec{v} = v_y\hat{x} + v_y\hat{y} = \langle v_x, v_y\rangle$$

$$\vec{a} = a_x\hat{x} + a_y\hat{y} = \langle a_x, a_y\rangle$$

FBD





$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

Newton 2nd

$$ma_x = F_{AIR,x} \qquad ma_y = F_{AIR,y} - F_{EARTH}$$

$$F_{AIR,x} = -D|\vec{v}|v_x \qquad F_{EARTH} = +mg$$

$$F_{AIR,y} = -D|\vec{v}|v_y$$

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$$ma_x = m\ddot{x} = -Dv_x\sqrt{v_x^2+v_y^2}$$

$$ma_y = m\ddot{y} = -Dv_y\sqrt{v_x^2+v_y^2} - mg$$

Coupled EoMs (cannot go further)

$$\dot{v}_x = -\tilde{D}\,v_x\sqrt{v_x^2+v_y^2} \qquad \text{with } \tilde{D} = D/m$$

$$\dot{v}_y = -\tilde{D}\,v_y\sqrt{v_x^2+v_y^2} - g$$

We need another approach since there's no analytical solution to these Differential Equations

$\Rightarrow$ cannot form,

$$f_1(v_x)\,dv_x = g_1(t)\,dt$$

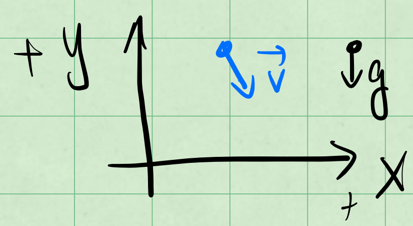
$$f_2(v_y)\,dv_y = g_2(t)\,dt$$

independent EoMs

so separation of variables is not possible

Example: Linear Drag in 2D

$$\vec{F}_{Lin} = -m\gamma\vec{v}$$



$$\vec{F}_{net} = \vec{F}_{lin} + \vec{F}_{EARTH}$$

$$ma_x = -m\gamma v_x$$

$$ma_y = -m\gamma v_y - mg$$

note v_x & v_y can be +, −, or, 0.
they are components

EOMs

$$\boxed{\dot{v}_x = -\gamma v_x \qquad \dot{v}_y = -\gamma v_y - g}$$

These are decoupled so we will try separation of variables

$$\dot{v}_x = \frac{dv_x}{dt} = -\gamma v_x$$

$$\frac{dv_x}{v_x} = -\gamma dt$$

Integrate!

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$$\int_{v_{0x}}^{v_x} \frac{dv_x'}{v_x'} = -\gamma \int_{t=0}^{t} dt'$$

$$\ln\left(v_x / v_{0x} \right) = -\gamma t$$

$$\boxed{v_x(t) = v_{0x} e^{-\gamma t}}$$

trajectory for v_x

$$\frac{dx}{dt} = v_x$$

$$\int_{x_0}^{x} dx' = \int_{t=0}^{t} v_{0x} e^{-\gamma t'} dt'$$

$$x - x_0 = -\frac{v_{0x}}{\gamma} \left(e^{-\gamma t'} \right)_0^t$$

$$x - x_0 = -\frac{v_{0x}}{\gamma} \left(e^{-\gamma t} - e^0 \right)$$

$$\boxed{x(t) = x_0 + \frac{1}{\gamma} v_{0x} \left(1 - e^{-\gamma t} \right)}$$

trajectory for x

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$$\dot{v}_y = -\gamma v_y - g$$

$$\frac{dv_y}{dt} = -\gamma v_y - g$$

$$\frac{dv_y}{\gamma v_y + g} = -dt \implies \frac{dv_y}{v_y + g/\gamma} = -\gamma dt$$

Integrate!!

$$\int \frac{dx}{a+x} = \ln(a+x) + C$$

$t=0$, $v_y = v_{0y}$

$$\int_{v_{0y}}^{v_y} \frac{dv_y'}{v_y' + g/\gamma} = \ln\left(\frac{g/\gamma + v_y}{g/\gamma + v_{0y}}\right) = -\gamma t$$

$$\left(\frac{g}{\gamma} + v_y\right) = \left(\frac{g}{\gamma} + v_{0y}\right)e^{-\gamma t}$$

$$\boxed{v_y(t) = \frac{g}{\gamma}\left(e^{-\gamma t} - 1\right) + v_{0y}e^{-\gamma t}}$$

trajectory for v_y.

$$\frac{dy}{dt} = v_y$$

$$\int_{y_0}^{y} dy' = y - y_0 = \int_{t=0}^{t} \left\{ \frac{g}{\gamma}\left(e^{-\gamma t'} - 1\right) + v_{0y} e^{-\gamma t'} \right\} dt'$$

$$y - y_0 = \frac{g}{\gamma}\left(\frac{-1}{\gamma} e^{-\gamma t'} - t'\right)\Big|_0^t + v_{0y}\left(-\frac{1}{\gamma} e^{-\gamma t'}\right)\Big|_0^t$$

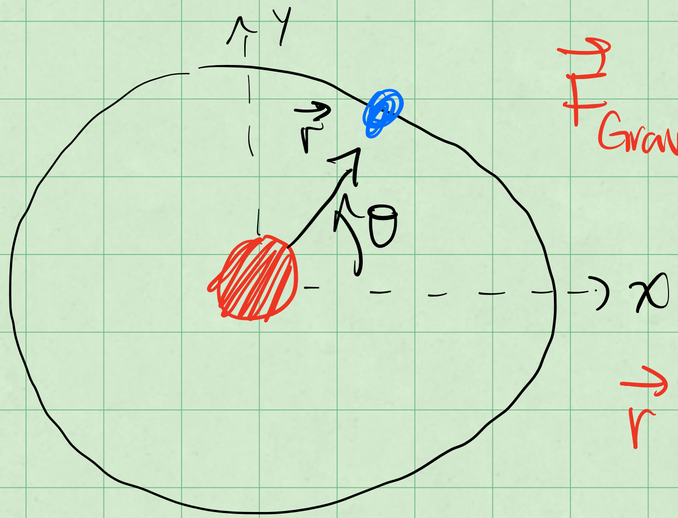
$$= \frac{g}{\gamma}\left(-\frac{1}{\gamma} e^{-\gamma t} - t + \frac{1}{\gamma}\right) + v_{0y}\left(-\frac{1}{\gamma} e^{-\gamma t} + \frac{1}{\gamma}\right)$$

$$y(t) = y_0 + \frac{v_{0y}}{\gamma} - \frac{v_{0y}}{\gamma} e^{-\gamma t} - \frac{g}{\gamma} t + \frac{g}{\gamma^2} - \frac{g}{\gamma^2} e^{-\gamma t}$$

$$\boxed{y(t) = y_0 - gt/\gamma + \frac{1}{\gamma}\left(v_{0y} + \frac{g}{\gamma}\right)\left(1 - e^{-\gamma t}\right)}$$

trajectory for y

Example: 2D Newtonian Gravitational Model (Sun - Earth)

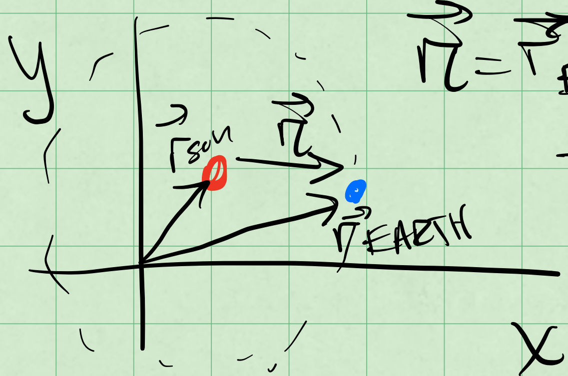


$$\vec{F}_{Grav} = \frac{-G m_1 m_2}{r^2}\hat{r}$$

$$\vec{r} = ?$$

$$\vec{r} = \vec{r}_{object1} - \vec{r}_{object2}$$

Move Sun off $\langle 0,0 \rangle$



$$\vec{r} = \vec{r}_{EARTH} - \vec{r}_{SUN}$$

Back to the Simplified 2D Model

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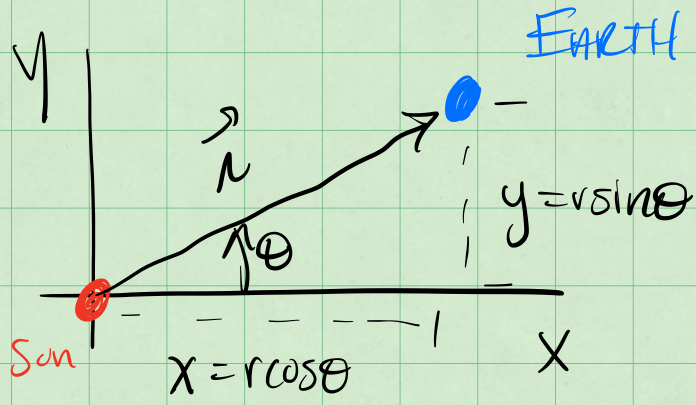


$M_\odot = 2\cdot 10^{30}\,kg$

$M_E = 6\cdot 10^{24}\,kg$

$G = 6.67\cdot 10^{-11}\,\frac{N}{kg^2}m^3$

$$\vec{F}_{Grav} = -G\frac{M_\odot M_E}{|\vec{r}|^3}\vec{r}$$

$|\vec{r}| = \sqrt{x^2+y^2}$
$= 1\,A.U$
$= 1.5\cdot 10^{11}\,m$

$$\vec{F}_{NEt} = \vec{F}_{Grav} = m\vec{a} = m\langle a_x\, a_y\rangle$$

$$F_x = \frac{-GM_\odot M_E x}{|r|^3} \qquad F_y = -\frac{GM_\odot M_E y}{|r|^3}$$

$$\boxed{\begin{aligned} a_x = \dot{v}_x = \ddot{x} &= \frac{-GM_\odot x}{(x^2+y^2)^{3/2}} \\ a_y = \dot{v}_y = \ddot{y} &= \frac{-GM_\odot y}{(x^2+y^2)^{3/2}} \end{aligned}}$$

EOMs
for Grav
(coupled)

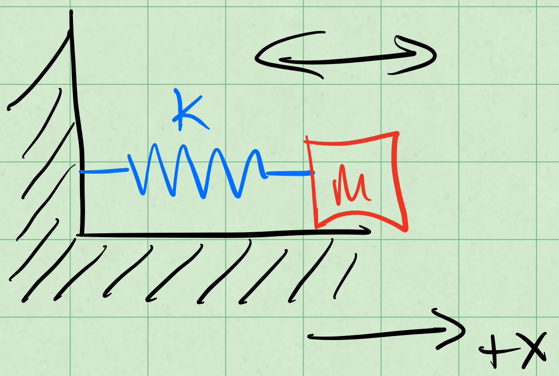
Trajectories?

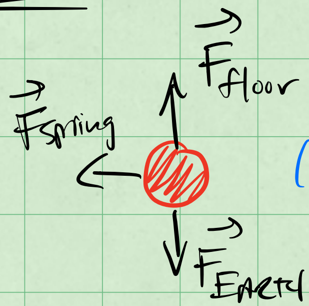
(1) Integrate EOMs

(2) solve coupled EOMs → techniques for this

(3) solve numerically

Example: Simple Harmonic Oscillator (1D)





(stretched right or compressed left)

$F_s = ks$ ← stretch $(x - L_0)$

relaxed length

if we measure displacement from equilibrium (L_0) then $F_s = kx$ direction opposite of stretch

$$F_x = ma_x = m\dot{v}_x = m\ddot{x} = -kx$$

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$$\boxed{m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} = -\frac{k}{m}x}$$

EoM of the SHO (Very important result)

Solutions?

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x \quad \text{where } \omega^2 = \frac{k}{m}$$

Try $x(t) = Ce^{i\omega t}$

$\dot{x}(t) = i\omega Ce^{i\omega t}$

$\ddot{x}(t) = -\omega^2 Ce^{i\omega t} = -\omega^2 x(t)$!!

$x(t) = Ce^{i\omega t}$ where $C = a + ib$ works!

So, maybe $\cos(\omega t)$ & $\sin(\omega t)$?

Try $x(t) = A\cos(\omega t) + B\sin(\omega t)$

$\dot{x}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$

$\ddot{x}(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

$= -\omega^2 x(t)$!!

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$$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t) \quad \text{with} \quad \omega^2 = k/m}$$

generic trajectory for SHO