

Day 24 - Resonance + Help Session

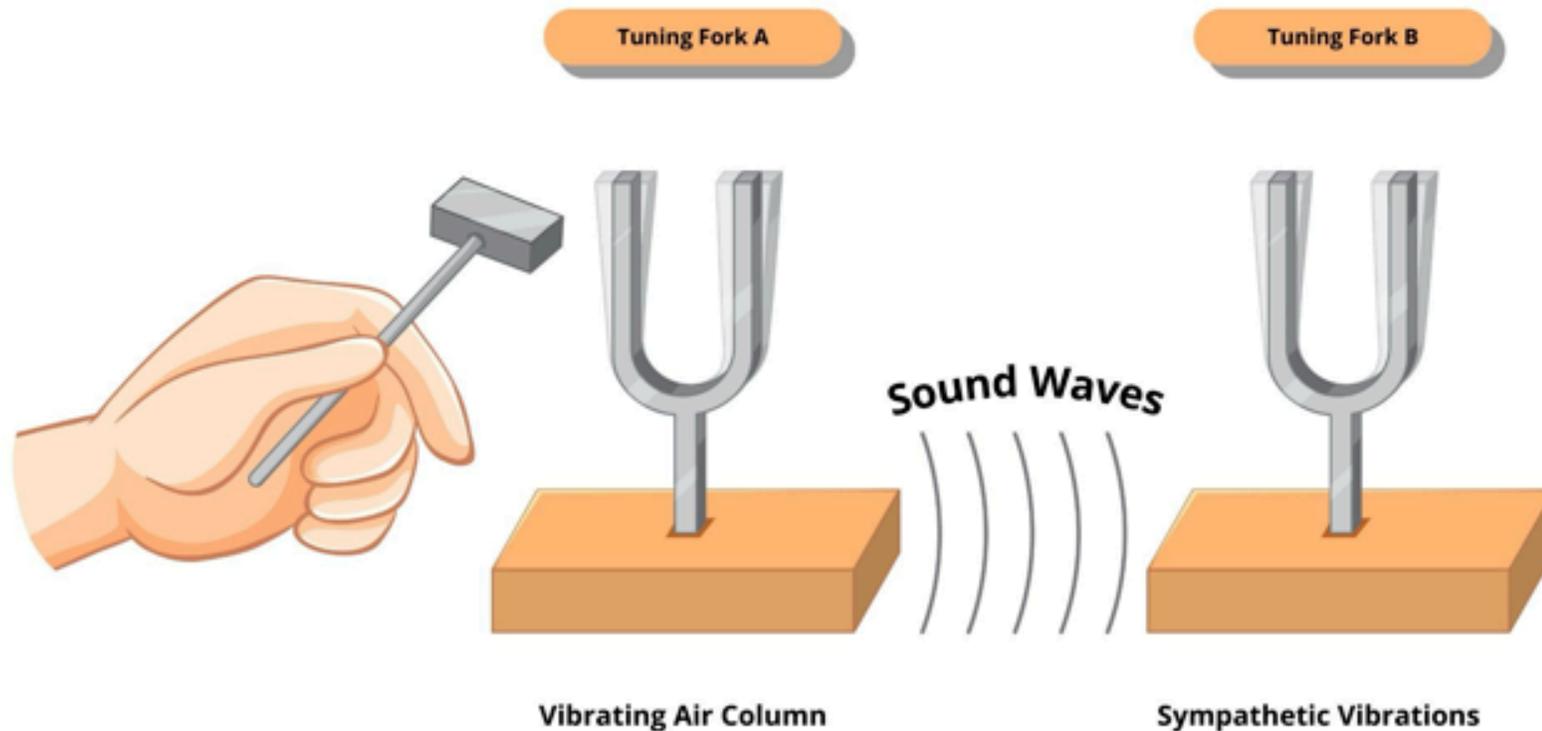


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Announcements

- Homework 5 extended to this Friday.
- Homework 6 due this Friday; may request extension up to a week if needed.
- Homework 7 posted, but will be **on time**
- **Midterm 2 is coming up!** Project question will be Ex 0; new turn in process; more details to come.
- Midterm 1 is graded; posted - Ex 0 is being graded now; will be posted with feedback by the end of the week.
- **Friday's Class:** Mihir will lead a homework help session - depending on your needs. *Vote at end of class.*

Reminders

We started to solve the forced harmonic oscillator equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

We examined the case of a sinusoidal driving force:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

There's a complimentary case where the driving force is a sine wave:

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$

Reminders

We combined the two equations into a complex equation using these identities:

$$z(t) = x(t) + iy(t)$$
$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

The resulting equation is:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

Notice that there's a homogeneous part (z_h) and a particular part (z_p).

$$\ddot{z}_h + 2\beta\dot{z}_h + \omega_0^2 z_h = 0$$

Reminders

The homogeneous part is the solution we've found before with the general solution:

$$z_h(t) = C_1 e^{rt} + C_2 e^{r^*t}$$

where $r = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$. In the case of a weakly damped oscillator ($\beta^2 < \omega_0^2$), we have:

$$z_h(t) = e^{-\beta t} \left(C_1 e^{-i\sqrt{\omega_0^2 - \beta^2}t} + C_2 e^{+i\sqrt{\omega_0^2 - \beta^2}t} \right)$$

These solutions die out as $t \rightarrow \infty$. They are called **transient solutions**.

Solving the particular part

The particular part is the solution to the driven harmonic oscillator equation:

$$\ddot{z}_p + 2\beta\dot{z}_p + \omega_0^2 z_p = f_0 e^{i\omega t}$$

Assume a sinusoidal solution (frequency, ω) of the form:

$$z_p(t) = C e^{i\omega t}$$

where C is a complex number. Then, we have:

$$\begin{aligned} -\omega^2 C e^{i\omega t} + 2i\beta\omega C e^{i\omega t} + \omega_0^2 C e^{i\omega t} &= f_0 e^{i\omega t} \\ (-\omega^2 + 2i\beta\omega + \omega_0^2) C &= f_0 \end{aligned}$$

Amplitude of the particular solution

$$C = \frac{f_0}{(\omega_0^2 - \omega^2 + 2i\beta\omega)}$$

We want to convert this to polar form:

$$C = Ae^{-i\delta}$$

where A and δ are real numbers. We use the complex form to compute the magnitude of the amplitude:

$$A^2 = C\bar{C} = \frac{f_0^2}{(\omega_0^2 - \omega^2 + 2i\beta\omega)(\omega_0^2 - \omega^2 - 2i\beta\omega)}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

Clicker Question 24-1

We found that the square amplitude of the driven harmonic oscillator is:

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

When is the amplitude of the driven oscillator maximized?

1. When the driving frequency (ω) is far from the natural frequency (ω_0)
2. When the driving frequency (ω) is close to the natural frequency (ω_0)
3. When the damping (2β) is weak
4. When the damping (2β) is strong
5. Some combination of the above

Finding the phase

With,

$$C = \frac{f_0}{(\omega_0^2 - \omega^2 + 2i\beta\omega)} = Ae^{-i\delta}$$

then we can compare the complex forms:

$$f_0 e^{i\delta} = A (\omega_0^2 - \omega^2 + 2i\beta\omega).$$

Both f_0 and A are real numbers, so the phase δ is the same phase as the complex number:

$$\delta = \tan^{-1} \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right)$$

The Particular Solution

Let's return to the particular solution:

$$z_p(t) = Ce^{i\omega t} = Ae^{-i\delta}e^{i\omega t} = Ae^{i(\omega t - \delta)}$$
$$z_p(t) = A \cos(\omega t - \delta) + iA \sin(\omega t - \delta)$$

So we get solutions to both driven oscillators:

$$x_p(t) = \operatorname{Re}(z_p(t)) = A \cos(\omega t - \delta)$$
$$y_p(t) = \operatorname{Im}(z_p(t)) = A \sin(\omega t - \delta)$$

These are the **steady-state solutions**.

They persist as $t \rightarrow \infty$ and oscillate at the driving frequency ω .

The Full Solution

$$x(t) = x_h(t) + x_p(t)$$

Here, $x_h(t)$ is the transient solution and $x_p(t)$ is the steady-state solution.

For weakly damped oscillators, the transient solution can be written in the form:

$$x_h(t) = A_{tr} e^{-\beta t} \cos(\sqrt{\omega_0^2 - \beta^2} t + \delta_{tr})$$

where A_{tr} and δ_{tr} are real numbers and are the amplitude and phase of the transient solution. Both are determined by the initial conditions.

$$x_p(t) = A \cos(\omega t - \delta)$$

where A and δ are real numbers and are the amplitude and phase of the steady-state solution.

The Full Solution

The transient plus the steady-state solution is the full solution:

$$x(t) = A_{tr} e^{-\beta t} \cos(\sqrt{\omega_0^2 - \beta^2} t + \delta_{tr}) + A \cos(\omega t - \delta)$$

As $t \rightarrow \infty$, the transient solution dies out and the steady-state solution persists.

$$x(t \rightarrow \infty) = A \cos(\omega t - \delta)$$

where

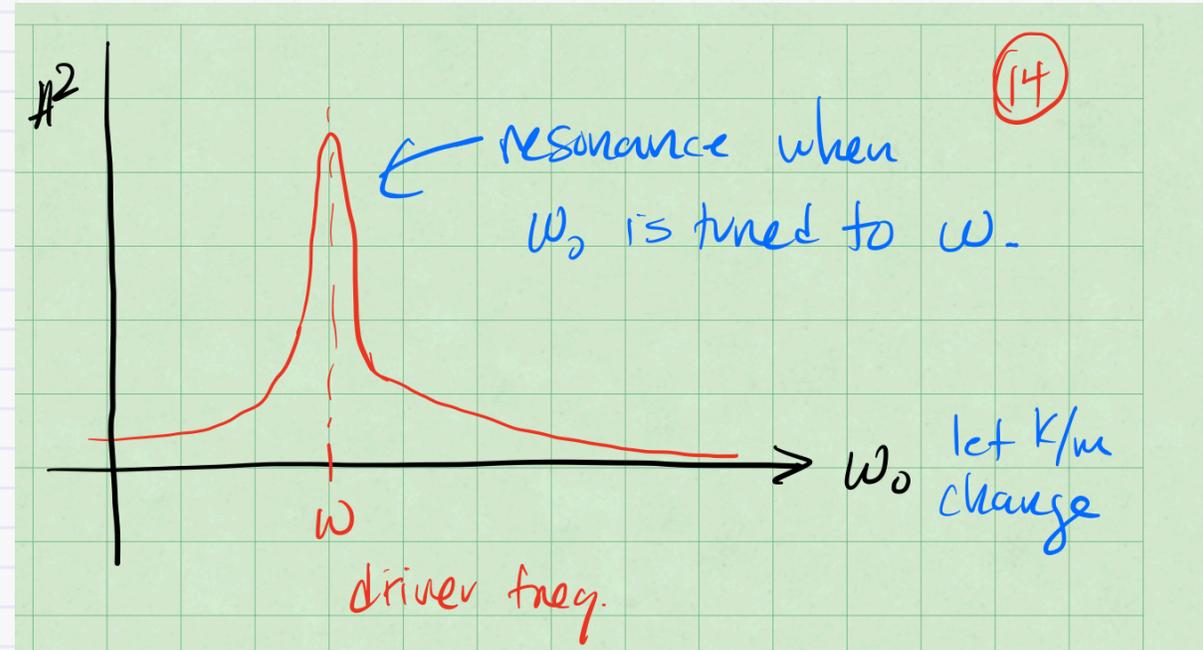
$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \quad \delta = \tan^{-1} \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right)$$

Resonance

The amplitude of the steady-state solution is:

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

We change ω_0 and observe how the amplitude changes.



Achieving resonance

The denominator of the equation controls the amplitude:

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$$

Case 1: Tune ω_0 to be close to ω . *Car Radio tuning*

With $\omega_0 = \omega$, the amplitude is:

$$A^2 = \frac{f_0^2}{4\beta^2\omega^2}$$

Achieving resonance

The denominator of the equation controls the amplitude:

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$$

Case 2: Tune ω to be close to ω_0 . *Pushing a swing*

Find the ω that maximizes the amplitude by taking the derivative with respect to ω :

$$\frac{d}{d\omega} ((\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2) = 0$$

$$2(\omega_0^2 - \omega^2)(-2\omega) + 8\beta^2\omega = 0$$

$$4\omega(\omega^2 - \omega_0^2 + 2\beta^2) = 0$$

$$\omega = 0 \quad \omega = \sqrt{\omega_0^2 - 2\beta^2}$$

Resonance is Not Just Classical

The amplitude of the driven oscillator near resonance:

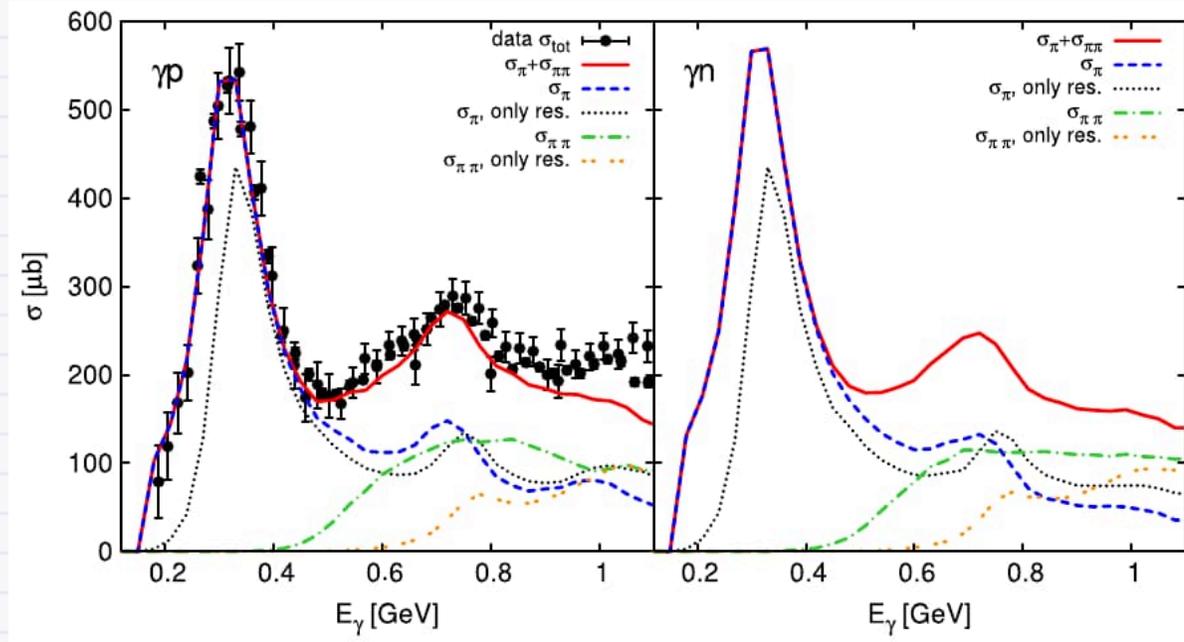
$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

In particle physics, the **Breit-Wigner formula** gives the cross section near a resonance:

$$\sigma(E) \propto \frac{\Gamma^2/4}{(E - M)^2 + \Gamma^2/4}$$

Same structure. The substitutions are: $M \leftrightarrow \omega_0$ (resonant mass/frequency) and $\Gamma \leftrightarrow 2\beta$ (decay width / damping). The **uncertainty principle** connects width to lifetime: $\Gamma = \hbar/\tau$.

The $\Delta(1232)$ Resonance



Cross section (probability) vs. energy for π^+ scattering off protons.

Seeing the $\Delta(1232)$

- Shoot π^+ (positive pions) at protons and vary the energy.
- Measure how often they interact: this gives the **cross section**.
- As the center-of-mass energy \sqrt{s} increases, the cross section suddenly **spikes**.
- This sharp peak occurs at

$$\sqrt{s} \approx 1232 \text{ MeV},$$

and is the signature of the $\Delta(1232)$ resonance.

What the Peak Tells Us

- The **position** of the peak gives the Δ mass:

$$M \approx 1232 \text{ MeV}/c^2.$$

- The **width** of the peak (how spread out it is in energy) is

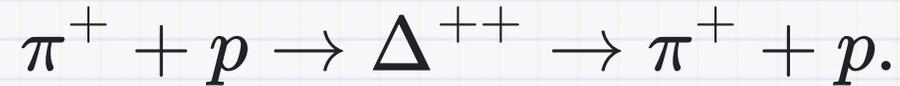
$$\Gamma \approx 117 \text{ MeV}.$$

- A narrow peak \Rightarrow longer-lived state; a broad peak \Rightarrow very short-lived state.
- From the width, we can estimate the lifetime:

$$\tau \approx \frac{\hbar}{\Gamma} \sim 5.6 \times 10^{-24} \text{ s}.$$

The Δ^{++} State

- Around the peak, the proton + π^+ system briefly forms a new state:



- The Δ^{++} has:
 - Mass $M \approx 1232 \text{ MeV}/c^2$,
 - Very short lifetime $\sim 10^{-23} \text{ s}$,
 - Main decay: $\Delta^{++} \rightarrow p + \pi^+$.
- We never “see” the Δ^{++} directly; we infer it from this enhanced scattering.

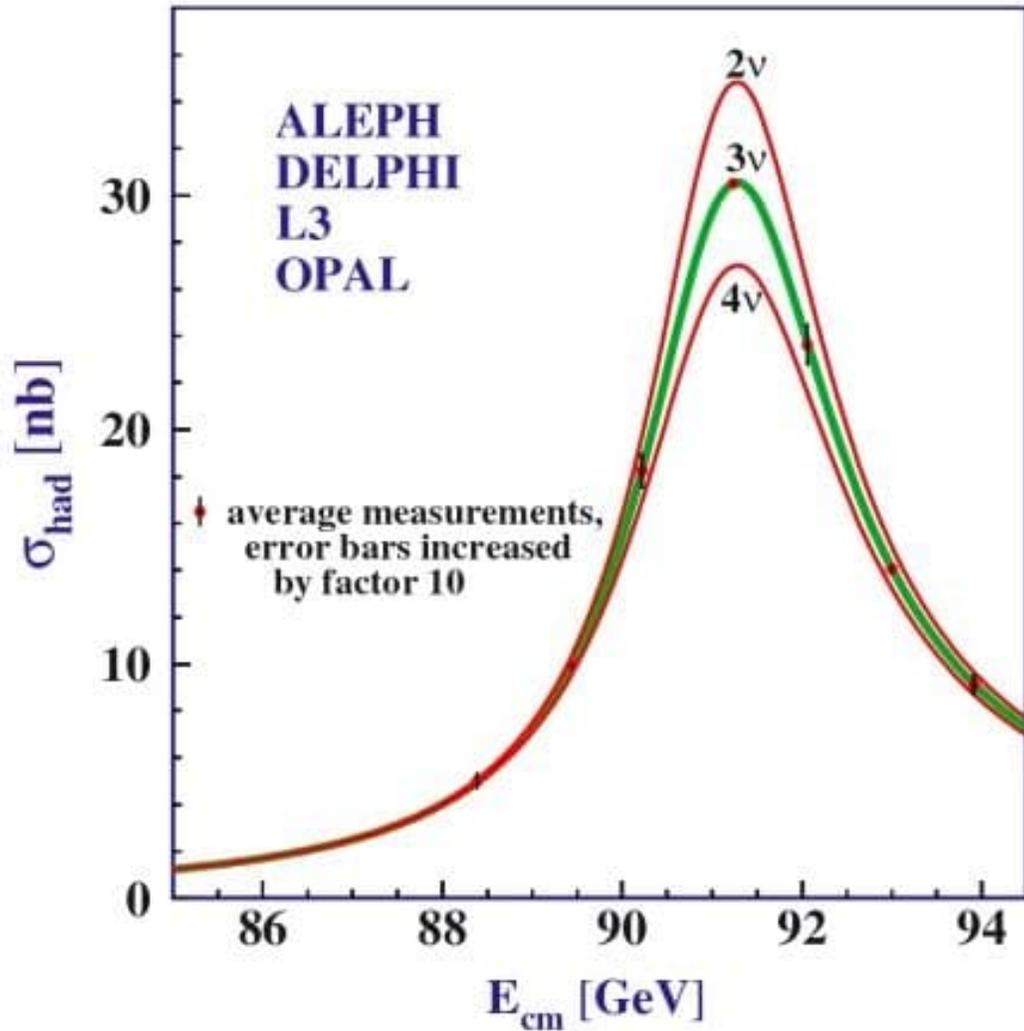
Resonance as a Driven Oscillator

- Think of the $\pi^+ p$ system like a **driven oscillator**:
 - Driving frequency \Leftrightarrow collision energy \sqrt{s} ,
 - Natural frequency $\omega_0 \Leftrightarrow$ resonance energy (1232 MeV),
 - Damping $\beta \Leftrightarrow$ decay width Γ .
- At $\sqrt{s} = 1232$ MeV, the system hits its **resonance** and the amplitude (cross section) is largest.
- Just as you can read off ω_0 and β from an amplitude-vs-frequency curve, you read off:
 - M from the **peak position**,
 - Γ from the **peak width**.

Why This Matters

- The $\Delta(1232)$ is an **excited state** of the nucleon, made of the same quarks arranged differently.
- It is a textbook example of how:
 - Short-lived states show up as **resonance peaks** in cross sections.
 - Lifetimes and internal structure can be extracted from scattering data.
- This is the same basic idea used later for heavier resonances, like the Z boson.

The Z Boson: Resonance - Counting the Universe



The Z Boson at LEP

- At the **LEP** collider at CERN, electrons and positrons were collided at different energies.
- When the total energy matched the **Z boson** mass, the production rate jumped.
- On a plot of event rate vs. energy, this shows up as a tall **bump** (a resonance).
- The center of this bump gives the Z mass:

$$M_Z \approx 91.19 \text{ GeV}/c^2$$

- LEP recorded about **17 million** Z decays.

Mass and Width of the Z

- The **peak position** of the bump tells us the Z mass:

$$M_Z \approx 91.19 \text{ GeV}/c^2$$

- The **width** of the bump tells us how quickly the Z decays:

$$\Gamma_Z \approx 2.495 \text{ GeV}$$

- Narrow bump \Rightarrow long-lived particle (few ways to decay).
- Broad bump \Rightarrow short-lived particle (**many** ways to decay).

What Determines the Width?

- The Z can decay in many ways:
 - Into quark–antiquark pairs (hadrons),
 - Into charged lepton pairs (e, μ, τ),
 - Into **neutrino–antineutrino** pairs.
- Each allowed decay channel adds a bit to the **total width** Γ_Z .
- More decay channels \Rightarrow the Z disappears faster \Rightarrow the bump gets broader.
- So the measured width Γ_Z encodes **all possible Z decays**.

Neutrinos and the Z Width

- Each neutrino type gives a channel:

$$Z \rightarrow \nu_i \bar{\nu}_i$$

- If there are N_ν different light neutrino types, the width scales with N_ν .
- Physicists predicted:
 - 2 neutrinos \Rightarrow smaller width,
 - 3 neutrinos \Rightarrow some width,
 - 4 neutrinos \Rightarrow noticeably **larger** width.
- By precisely measuring the Z lineshape, LEP could **count** how many neutrinos contribute.

What LEP Found

- LEP measured the Z resonance shape extremely precisely.
- The observed total width:

$$\Gamma_Z \approx 2.495 \text{ GeV}$$

matches the prediction for **3** light neutrino types.

- A fourth light neutrino would have made the peak **clearly wider** than what LEP saw.
- Conclusion:
 - There are **three** light neutrino generations.
 - This matches the three known generations of quarks and leptons.

Big Idea

- A resonance peak (like the Z) is not just a pretty bump:
 - Peak position \Rightarrow particle **mass**.
 - Width \Rightarrow particle **lifetime** and number of decay channels.
- By studying the **shape** of the Z boson resonance, LEP physicists learned:
 - The mass and lifetime of the Z.
 - That nature has **three** generations of light neutrinos.

Resonance physics let us count how many generations of matter our universe has.

HW6 Exercise 1: Morse Potential as an SHO

If the potential has a local minimum, we can often find SHO approximation for that potential near the local minimum.

The **Morse potential** is a convenient model for the potential energy of a diatomic molecule. The potential is a radial one and thus one-dimensional. It is given by,

$$U(r) = A \left[\left(e^{(R-r)/S} - 1 \right)^2 - 1 \right]$$

where the distance between the centers of the two atoms is r , and the constants A , R , and S are all positive. Here $S \ll R$.

- 1a. Sketch (or plot) the potential as a function of r .

HW6 Exercise 1: Morse Potential as an SHO

$$U(r) = A \left[\left(e^{(R-r)/S} - 1 \right)^2 - 1 \right]$$

- 1b. Find the equilibrium position of the potential, i.e. the position where the potential is at a minimum. We will call this r_e .
- 1c. Rewrite the potential in terms of the displacement from equilibrium, $r = r_e + x$. Expand the potential to second order in x .
- 1d. Find the effective spring constant, k , for the potential near the minimum. What is the frequency of small oscillations about the minimum?

HW6 Exercise 3: Toy Potential

Consider a toy potential of the form,

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

where U_0 , R , and λ are all positive constants and the domain of the potential is $0 < r < \infty$.

- 3a. Sketch (or plot) the potential as a function of r .

HW6 Exercise 3: Toy Potential

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

- 3b. Find the equilibrium position of the potential, i.e. the position where the potential is at a minimum. We will call this r_e .
- 3c. Rewrite the potential in terms of the displacement from equilibrium, $r = r_e + x$. Expand the potential to second order in x . What is the effective spring constant, k , for the potential near the minimum? What is the frequency of small oscillations about the minimum?