Day 14 - Potential Energy and Stability

Mexican Hat/Sombrero Potential

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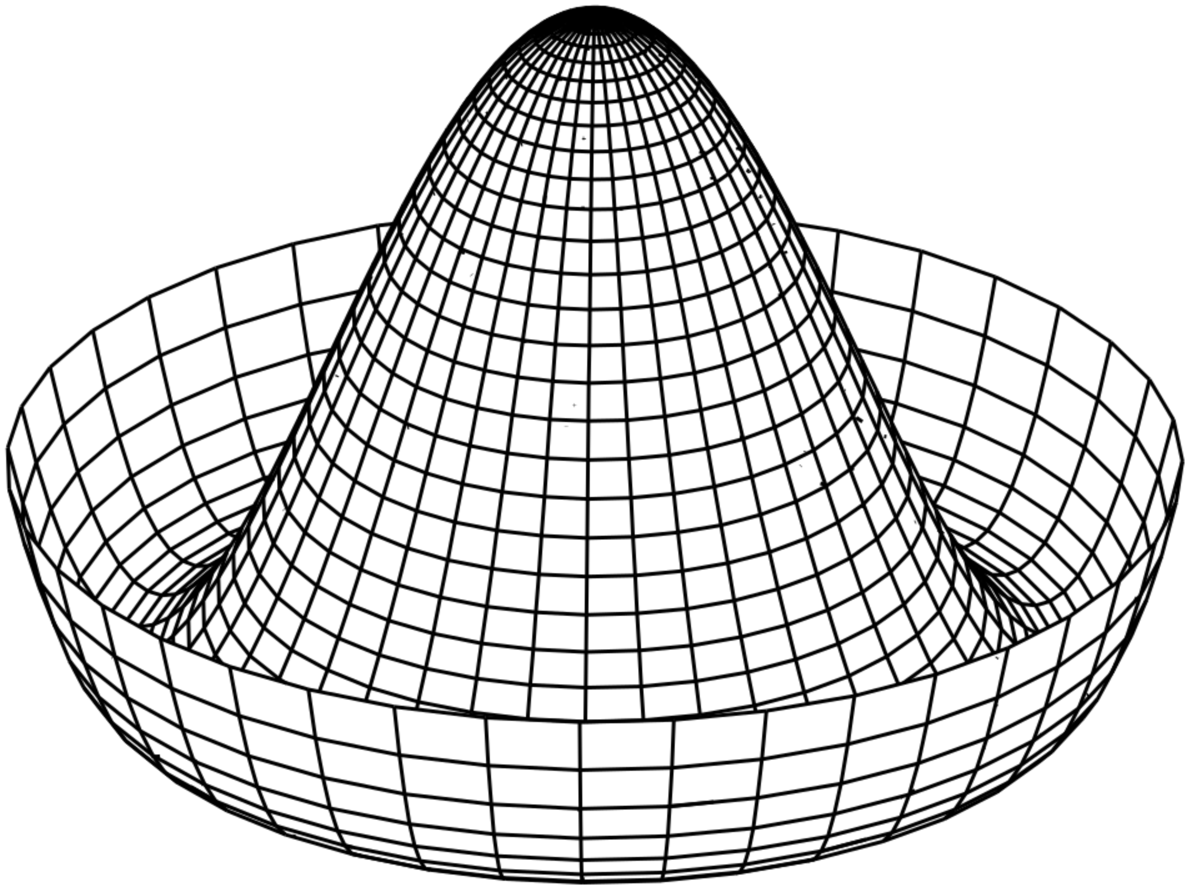
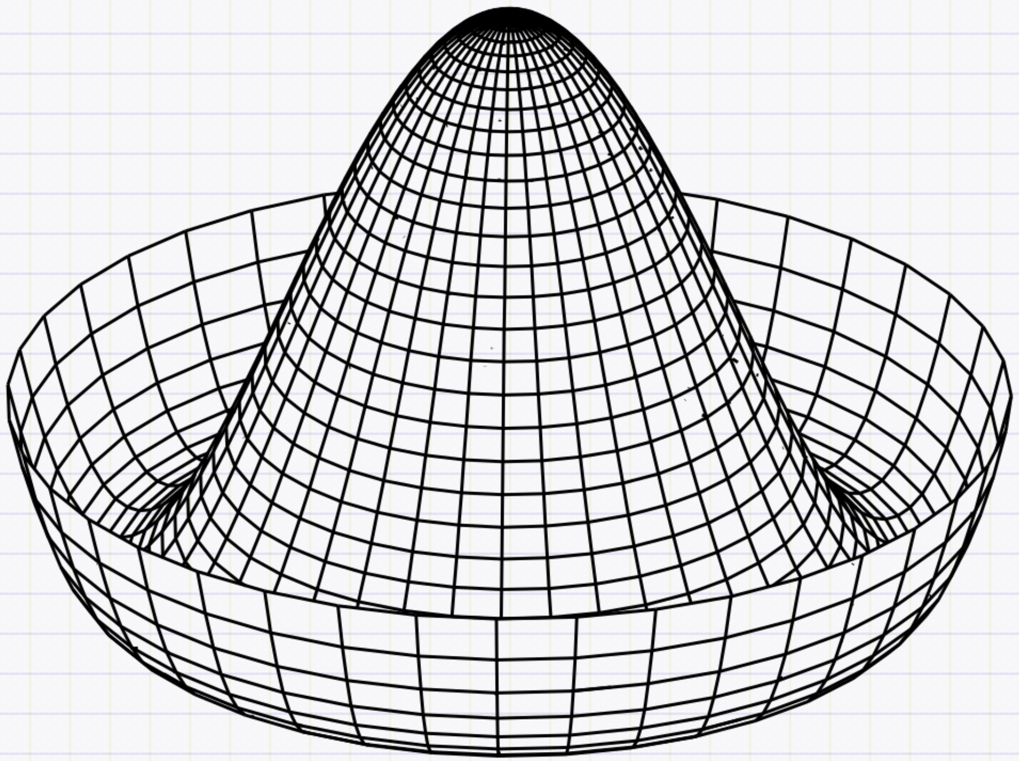


Mexican Hat Potential

$$V(\phi) = -5|\phi|^2 + |\phi|^4$$

- Spontaneous Symmetry Breaking (Jeffery Goldstone, 1961)
- Unstable vacuum state at $\phi = 0$
  - Peak of the hat
- Infinite number of stable minima
  - $\phi = \sqrt{5/2}e^{i\phi}$



Welcome Prof. Rachel Henderson!

Announcements

- Midterm 1 is available today (Due 25 Feb; late 27 Feb)
- DC will say more about this on Wednesday, but:
  - You may work in larger groups, but solutions are submitted like homework (max 3 group members) **on Gradescope**
  - Exercise 0 is for project planning; and can be submitted individually or as a *different* group **on D2l**

This Week's Goals

- Understand the concept of potential energy
- Determine the equilibrium points of a system using potential energy
- Analyze the stability of equilibrium points
- Define and begin to apply conservation of linear and angular momentum

Reminder: The Gradient Operator ∇

∇ is a vector operator. In Cartesian coordinates:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Acting on a scalar function $f(x, y, z)$ produces a vector:

$$\nabla f(x, y, z) = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Reminder: The Gradient Operator ∇

∇ can act on vector field (function), $\mathbf{F}(x, y, z)$ with both dot and cross products.

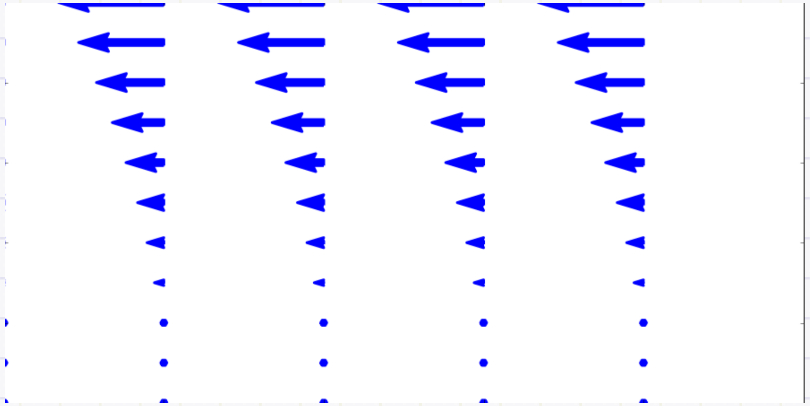
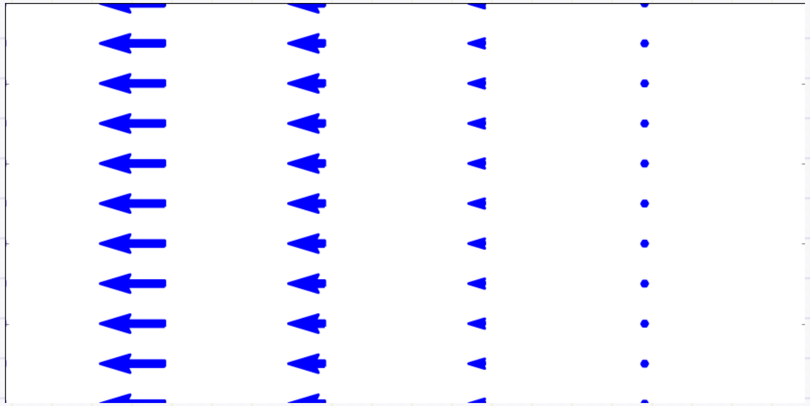
Divergence (Scalar Product) - How does the vector field change in the direction of the vector?

$$\nabla \cdot \mathbf{F}(x, y, z) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_x, F_y, F_z \rangle$$

$$\nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Clicker Question 14-1a

Which of the following fields have no divergence?

A.  B. 

1. A
2. B
3. Both A and B
4. Neither A nor B

Reminder: The Gradient Operator ∇

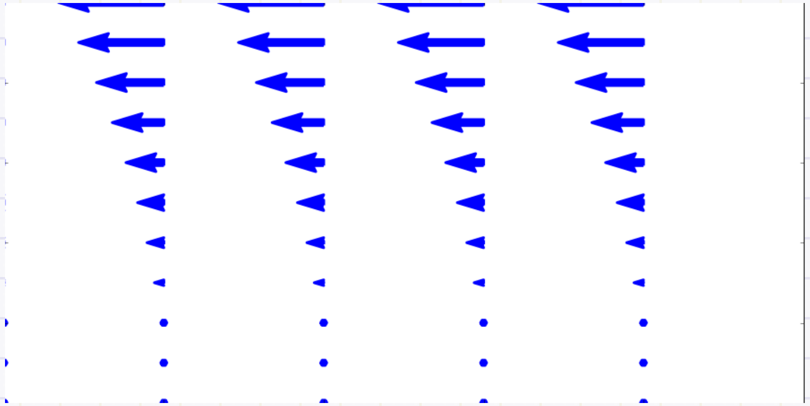
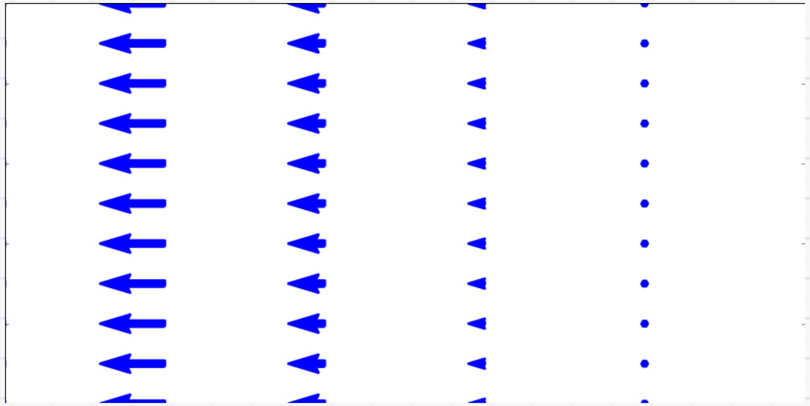
Curl (Vector Product) - How does the vector field change in the direction perpendicular to the vector?

$$\nabla \times \mathbf{F}(x, y, z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla \times \mathbf{F}(x, y, z) = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

Clicker Question 14-1b

Which of the following fields have no curl?

A.  B. 

1. A
2. B
3. Both A and B
4. Neither A nor B

Clicker Question 14-1c

Consider a vector field with zero curl: $\nabla \times \vec{F} = 0$. Which of the following statements is true?

1. The field is conservative
2. $\int \nabla \times \vec{F} \cdot d\vec{A} = 0$
3. $\oint \vec{F} \cdot d\vec{r} \neq 0$
4. $\vec{F}$ is the gradient of some scalar function, e.g., $\vec{F} = -\nabla U$
5. Some combination of the above

Reminders: Conservative Forces

- Conservative forces are those with zero curl

$$\nabla \times \vec{F} = 0$$

- The work done by a conservative force is path-independent; on a closed path, the work done is zero

$$\oint \vec{F} \cdot d\vec{r} = 0$$

- The force can be written as the gradient of a scalar potential energy function

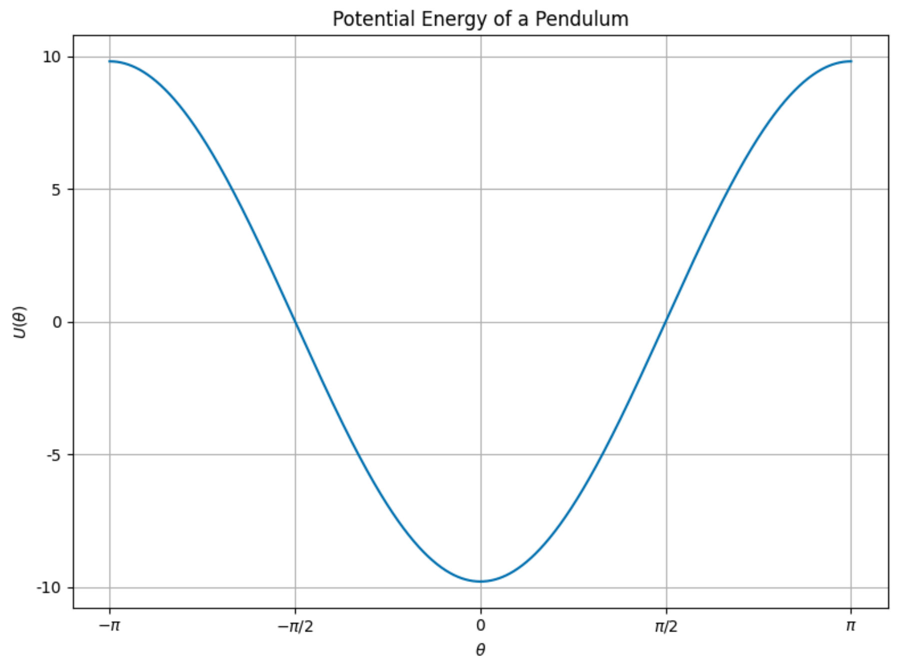
$$\vec{F} = -\nabla U$$

Clicker Question 14-2

Here's the graph of the potential energy function $U(x)$ for a pendulum.

What can you say about the equilibrium points? There is/are:

1. One stable point
2. Two stable points
3. One stable and one unstable point
4. Two unstable and one stable point



Clicker Question 14-3

Here's a potential energy function $U(x)$ for a pendulum:

$$U(\phi) = -mgL\cos(\phi) + U_0$$

1. Find the equilibrium points (ϕ^*) of the pendulum by setting:

$$\frac{dU(\phi^*)}{d\phi} = 0.$$

2. Characterize the stability of the equilibrium points (ϕ^*) by examining the second derivative:

$$\frac{d^2U(\phi^*)}{d\phi^2} > 0? \qquad \frac{d^2U(\phi^*)}{d\phi^2} < 0?$$

Click when done.

Clicker Question 14-4

A double-well potential energy function $U(x)$ is given by

$$U(x) = -\frac{1}{2}kx^2 + \frac{1}{4}kx^4.$$

We assume we have scaled the potential energy so that all the units are consistent.

How many equilibrium points does this system have?

1. 1
2. 2
3. 3
4. 4

Clicker Question 14-5

A double-well potential energy function $U(x)$ is given by

$$U(x) = -\frac{1}{2}kx^2 + \frac{1}{4}kx^4.$$

1. Find the equilibrium points (x^*) of the pendulum by setting:

$$\frac{dU(x^*)}{dx} = 0.$$

2. Characterize the stability of the equilibrium points (x^*):

$$\frac{d^2U(x^*)}{dx^2} > 0? \qquad \frac{d^2U(x^*)}{dx^2} < 0?$$

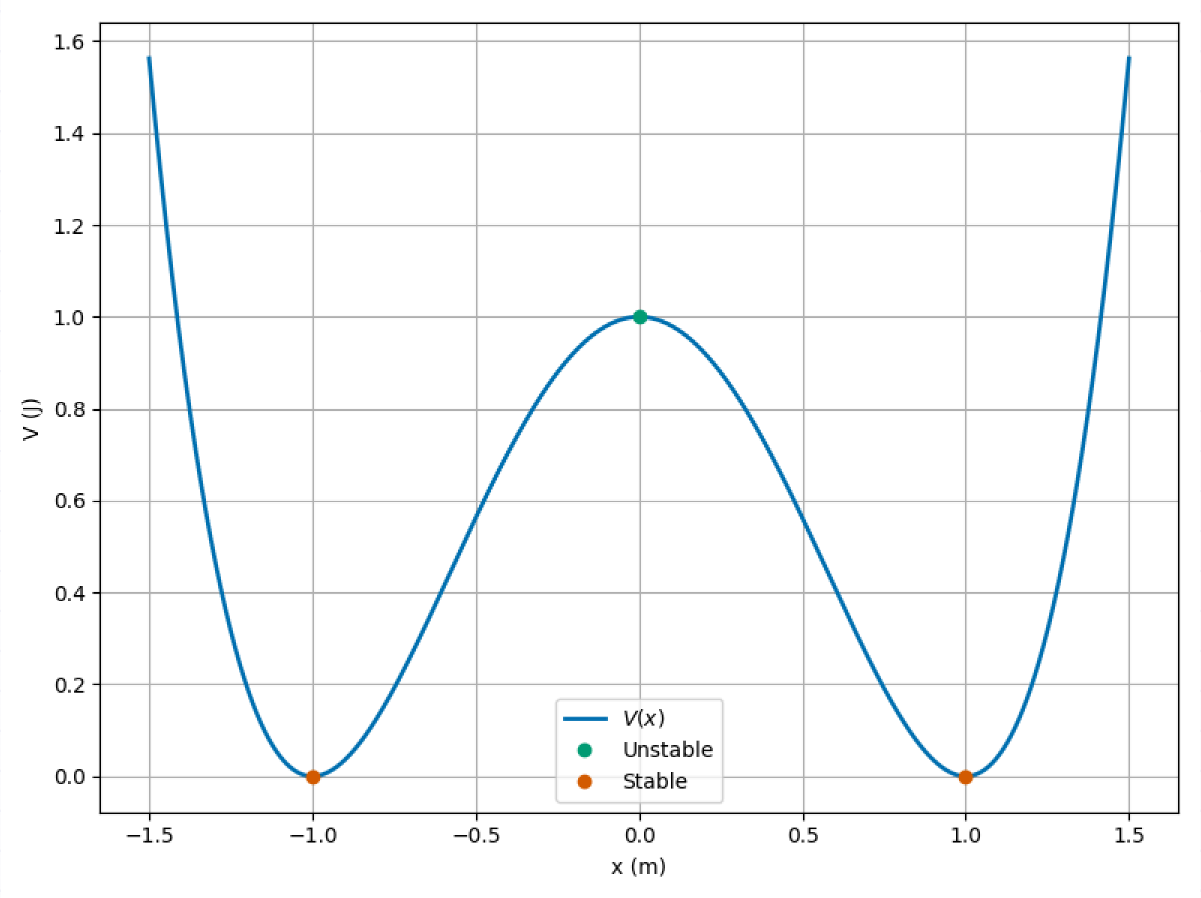
Click when done.

Clicker Question 14-6

Here's a graph of the potential energy function $U(x)$ for a double-well potential.

Describe the motion of a particle with the total energy, $E =$

1. $0.4\,\mathrm{J}$, $<$ barrier height
2. $1.2\,\mathrm{J}$, $>$ barrier height
3. $1.0\,\mathrm{J}$, $=$ barrier height



Click when done.