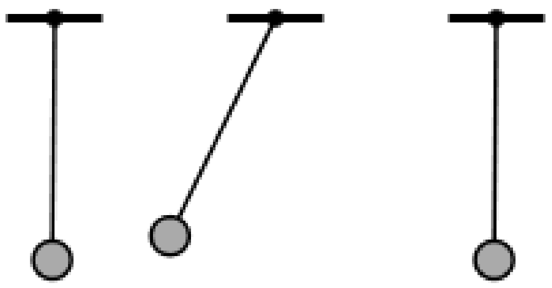
Day 23 - Driven Oscillations

Resonance in a driven pendulum system. $\longrightarrow$

Source: Wikipedia



Welcome Richard Hallstein!

Announcements

- HW 5 given blanket extension to this Friday.
- HW 6 still due this Friday, but will be extended if needed.
- DC will return Wednesday
- Mihir will lead workshop session for HW 6 on Friday

Reminders

We solved the damped harmonic oscillator equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

We chose a solution (**ansatz**) of the form

$$x(t) = C_1 e^{rt} + C_2 e^{rt}$$

and computed the roots of the characteristic equation:

$$r^2 + 2\beta r + \omega_0^2 = 0$$

We found the roots to be:

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Weak Damping

We found that when $\beta^2 < \omega_0^2$, the roots are complex:

$$r = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

This means that the solution is oscillatory:

$$x(t) = e^{-\beta t}\left(C_1 \cos(\sqrt{\omega_0^2 - \beta^2}t) + C_2 \sin(\sqrt{\omega_0^2 - \beta^2}t)\right)$$

The solution is a damped oscillation with frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

Strong Damping

When $\beta^2 > \omega_0^2$, the roots are real:

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

This means that the solution is not oscillatory:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where $r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} < 0$ and $r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2} < 0$.

The solution is the sum of two exponentials with different decay rates.

Critical Damping

When $\beta^2 = \omega_0^2$, the roots are real and equal (repeated roots):

$$r = -\beta$$

This means that the solution is not oscillatory, but also that our ansatz is not sufficient. The correct form of the solution is:

$$x(t) = (C_1 + C_2 t)e^{-\beta t}$$

In most cases, we will work with weak damping.

Clicker Question 24-1

What do we expect the phase space diagram (x vs $\dot{x}$) to look like for a weakly damped harmonic oscillator?

1. A set of ellipses
2. A set of spirals
3. Depends on how weak the damping is
4. Depends on the total energy
5. More than one of the above

Clicker Question 24-3

The driven harmonic oscillator equation is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

with $w_0^2 = k/m$ and $2\beta = b/m$. What is the dimension of the driving force $f(t)$?

1. Force (Newtons, N)
2. Force per unit second (N/s)
3. Force per unit length (N/m)
4. Force per unit mass (N/kg)

Clicker Question 24-4

The driven harmonic oscillator equation is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t).$$

This ODE is a _________ differential equation.

1. linear
2. nonlinear
3. first-order
4. second-order
5. more than one of the above

Example: Sinusoidal Driving Force

Let $f(t) = f_0 \cos(\omega t)$, so that the driven harmonic oscillator equation is:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

Note: $\omega \neq \omega_0$

Note that if the driving follows a sine wave, then we have:

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$

Interesting, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, let try to work with $z(t) = x(t) + iy(t)$.

Clicker Question 24-5

We found that the square amplitude of the driven harmonic oscillator is:

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

When is the amplitude of the driven oscillator maximized?

1. When the driving frequency (ω) is far from the natural frequency (ω_0)
2. When the driving frequency (ω) is close to the natural frequency (ω_0)
3. When the damping (2β) is weak
4. When the damping (2β) is strong
5. Some combination of the above