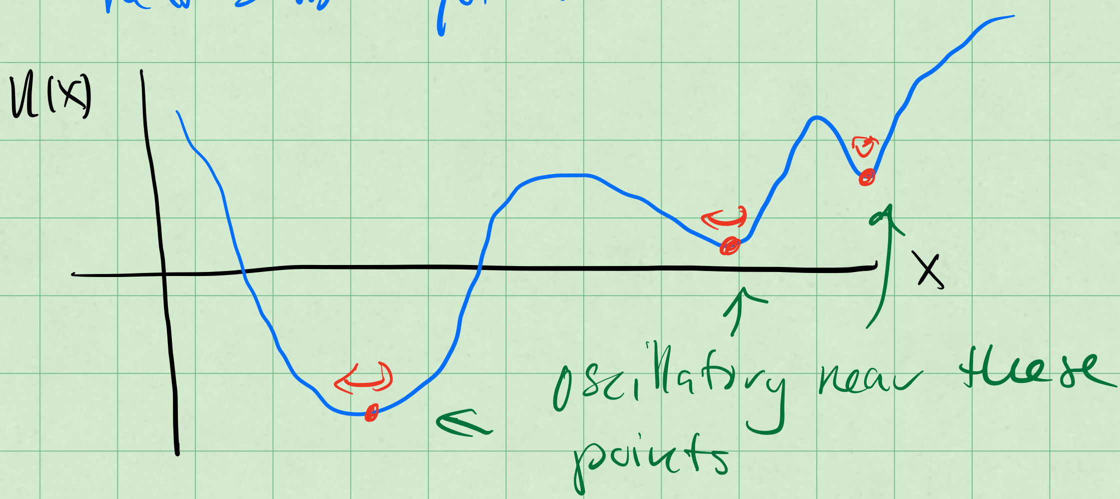
CW7 - Oscillations

We continue our study of physics phenomena to behavior that is recurrent, i.e. it repeats in some fashion.

We have met this behavior many times. This is because we often look at the behavior of systems just a little way from equilibrium.

Systems with local equilibria will have oscillatory behavior in a region near stable equilibria.



We have seen why this is a frequent occurence. For a given stable pt, $x=a$, we expand $U(x)$ around that point, (2)

$$U(x) \simeq U(a) + \underbrace{\left.\frac{dU}{dx}\right|_{x=a}(x-a)}_{\substack{0 \text{ by defn} \\ x=a \text{ is a local min}}} + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=a}(x-a)^2$$

$$U(x) \simeq U(a) + \underbrace{\frac{1}{2}\left.\frac{d^2U}{dx}\right|_{x=a}}_{\substack{\text{this is just the concavity a } x=a \\ \text{Call it 'k'}}}(x-a)^2$$

near $x=a$,

$$U(x) = U(a) + \frac{1}{2}k(x-a)^2$$

But only changes in U matter,

$$\Delta U = \frac{1}{2}k(x_f-a)^2 - \frac{1}{2}k(x_i-a)^2$$

$$\Delta U = U_f - U_i$$

The Exact Solutions

This potential $U = \frac{1}{2}kx^2$ gives a force that is a restoring one,

$$F = -kx$$

and thus, $m\ddot{x} = -kx$

so that,

$$\boxed{\ddot{x} = \frac{-k}{m}x}$$

Diffy Q of the SHO $\longrightarrow$

The oscillation frequency, ω, is related to the coupling constant k/m. Namely,

$$\boxed{\omega^2 = k/m}$$

frequency of SHO with spring mass system

You've likely seen solutions like,

$x(t) = A_1\cos(\omega t) + A_2\sin(\omega t)$ $\quad$ $x(t) = A\cos(\omega t + \delta)$

But we are going to find more utility in the complex form. 4

Let's try the guess (the ansatz)

$$x(t) = e^{i\omega t} \quad \text{where } i^2 = -1$$

$$\dot{x}(t) = i\omega e^{i\omega t}$$

$$\ddot{x}(t) = (i\omega)(i\omega)\, e^{i\omega t} = -\omega^2 e^{i\omega t}$$

or $\ddot{x}(t) = -\omega^2 x(t)$ ✓

Notice that $x(t) = e^{-i\omega t}$ also works. So will any linear combination.

In general,

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

where C_1 & C_2 can be complex.

2^{nd} order ODE $\Rightarrow$ 2 arbitrary constants

It must be that $x(t)$ be real so this will tell us about C_1 & C_2.

To connect complex exponentials to sine and cosine we note the Euler relationship,

$$\boxed{e^{i\theta} = \sin\theta + i\cos\theta}$$

So that

$$e^{\pm i\omega t} = \sin(\omega t) \pm i\cos(\omega t)$$

with

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$= \underbrace{(C_1 + C_2)}_{B_1} \sin(\omega t) + i\underbrace{(C_1 - C_2)}_{B_2} \cos(\omega t)$$

$$x(t) = B_1 \sin(\omega t) + B_2 \cos(\omega t)$$

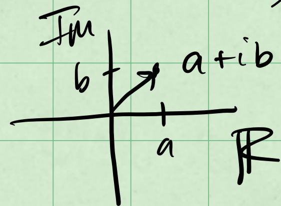
Great! so B_1 & B_2 better be real. What does that mean?

Rewritten,

$$C_1 = \frac{1}{2}(B_1 - iB_2) \qquad C_2 = \frac{1}{2}(B_1 + iB_2)$$

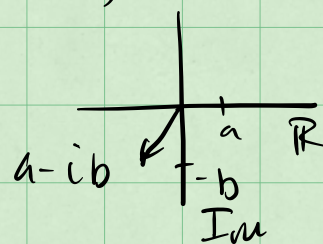
Notice that C_1 & C_2 are "complex conjugates" for a complex number,

$$z = a + ib$$



the complex conjugate is,

$$z^* = a - ib$$



so that,

$$z z^* = (a+ib)(a-ib)$$

$$zz^* = a^2 - iab + iab + ib(-ib)$$

$$zz^* = a^2 + b^2 \quad \text{is real.} \Rightarrow |z| = \sqrt{zz^*}$$

So, $\quad x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

$C_2 = C_1^*$ or,

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$$x(t) = C_1 e^{i\omega t} + C_1^* e^{-i\omega t}$$

Note also, $z + z^* = a + ib + a - ib = 2a$

$z + z^* = 2\,\mathrm{Re}\, z$ ← the Real part of z.

$\mathrm{Re}\, z = \mathrm{Re}\, z^* = a$

$\mathrm{Im}\, z = -\mathrm{Im}\, z^* = b$ ← the Imaginary part of a complex # is Real.

$$x(t) = C_1 e^{i\omega t} + C_1^* e^{-i\omega t}$$

$$= C_1 e^{i\omega t} + \left(C_1 e^{i\omega t}\right)^* = 2\,\mathrm{Re}\, C_1 e^{i\omega t}$$

$$x(t) = 2\,\mathrm{Re}\left(C_1 e^{i\omega t}\right)$$

let $C = 2C_1$

$$x(t) = \mathrm{Re}\left(C e^{i\omega t}\right) = A\cos(\omega t - \delta)$$

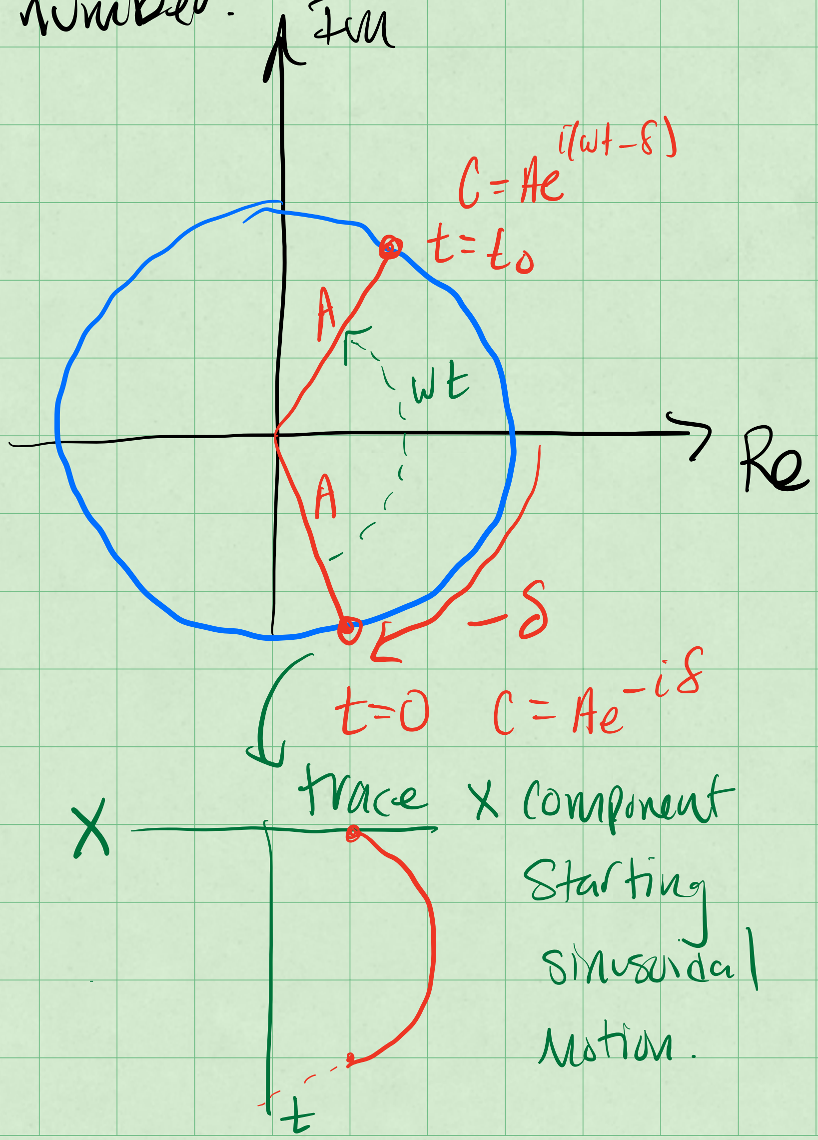
$$= \mathrm{Re}\left(A e^{i(\omega t - \delta)}\right)$$ so that

$$C = A e^{-i\delta}$$

But most importantly from a conceptual point is, the solution,

$$x(t) = \mathrm{Re}\left(Ae^{i(\omega t-\delta)}\right),$$

gives a projection of a rotating complex number.



Damping Oscillations

The first complication we can add is a bit of damping. Here we expect energy to be lost over time, but how?

Let's start with the equation of motion,

$$m\ddot{x} + b\dot{x} + kx = 0$$

Let's make a few simplifications,

$$\ddot{x} + \frac{b}{m}\dot{x} + k/m\, x \qquad \text{let} \quad \omega_0 = \sqrt{k/m} \;\leftarrow \text{natural freq.}$$

$$\text{and} \quad \frac{b}{m} = 2\beta$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

damping coeff.

This is a linear ODE so if we find a solution, and it fits our conditions for the system → it is guarenteed to be the only one. ⇐ Thank you, uniqueness theorem!

So, let's guess $x(t) = e^{rt}$ as we have seen that form work before.

$\dot{x}(t) = re^{rt}$ $\ddot{x}(t) = r^2 e^{rt}$

with

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$r^2 \cancel{e^{rt}} + 2\beta r \cancel{e^{rt}} + \omega_0^2 \cancel{e^{rt}} = 0$$

$$r^2 + 2\beta r + \omega_0^2 = 0$$ Auxilary Eqn.

if we can solve for r^2 we find the general solutions!

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$$

$$r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

The general solution is the linear superposition.

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$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\boxed{x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)}$$

Let's consider a few cases,

No damping? $\beta = 0$

then

$$x(t) = e^0\left(C_1 e^{\sqrt{-\omega_0^2}\,t} + C_2 e^{-\sqrt{-\omega_0^2}\,t}\right)$$

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$ like before

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Weak Damping $\beta < \omega_0$ (Technically weak is $\beta \ll \omega_0$, but whatever)

$$\beta^2 - \omega_0^2 < 0$$

So,

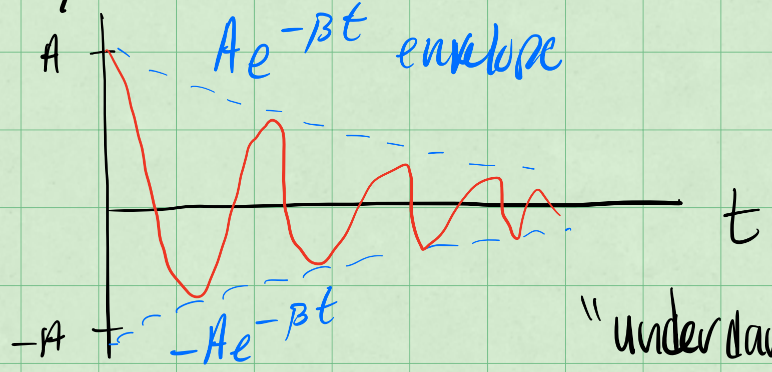
$$\sqrt{\beta^2-\omega_0^2} = i\sqrt{\omega_0^2-\beta^2} = i\omega_1$$ where ω_1 is real

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

So that,

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

Equivalent form $x(t) = Ae^{-\beta t}\cos(\omega t - \delta)$



better term

"underdamped" motion

Strong Damping $(\beta > \omega_0)$ $\left(\begin{array}{l}\text{Again tech.}\\ \beta >> \omega_0\end{array}\right)$

$\beta^2 - \omega_0^2 > 0$ so that

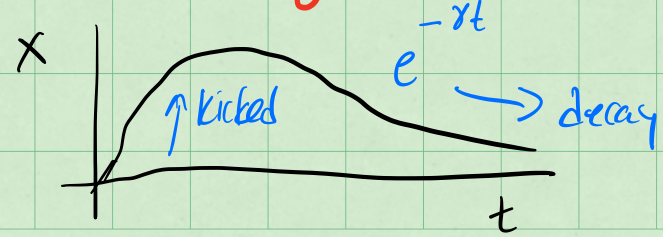
$\sqrt{\beta^2 - \omega_0^2}$ is Real

$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}$$

exponential decay as $t \to \infty$

better word.

"overdamped" motion



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What about $\beta = \omega_0$? Critical Damping

$\beta^2 - \omega_0^2 = 0$ So that,

$X(t) = e^{-\beta t}$? only??

No. our guess $x(t) = e^{rt}$ is ok for both solutions unless $\beta = \omega_0$ then

$X(t) = te^{-\beta t}$ is also needed.

$$\boxed{x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}}$$

Critically damped.