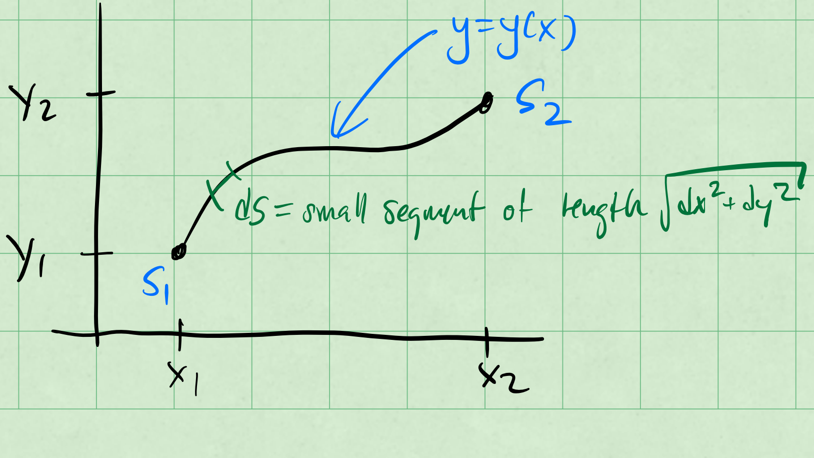
Calculus of Variations

- as we will use it the calculus of variations focuses on finding (the conditions of) extrema for quantities that can be expressed as an integral.
- this might seem odd but turns out to be an interesting way to develop an equivalent formulation of mechanics.

Canonical Conceptualization

Using Calculus of variations we can show the shortest distance between two points is a line.

Consider a general path in 2D



The length of the path is the integral from S_1 to S_2 2

$$\ell = \int_{S_1}^{S_2} ds$$

← we want to minimize ℓ, which is minimizing the integral.

Let's write ds in terms of dx, dy, and $y(x)$.

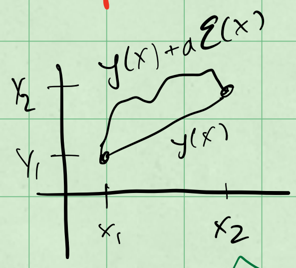
$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + y'(x)^2}$$

so that,

$$\ell = \int_{x_1}^{x_2} dx \sqrt{1 + [y'(x)]^2}$$

$y'(x)$ defines the path

To go further, we need to posit an incorrect solution,



$$Y(x) = \underbrace{y(x)}_{\text{correct}} + \underbrace{\alpha\mathcal{E}(x)}_{\text{error to minimize.}}$$

$Y(x_1) = y_1$ $\quad$ $Y(x_2) = y_2$

And we need to investigate what happens in general, 3

Take a function that is being integrated,

$$f(y(x), y'(x), x)$$

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

assume $y(x)$ minimizes S such that any function $Y(x) = y(x) + \underbrace{\alpha\varepsilon(x)}_{\text{error term}}$

produces a larger integral,

$$\int_{x_1}^{x_2} f(Y(x), Y'(x), x)\, dx > \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

Thus $S(\alpha=0)$ is a minimum, what conditions does that produce?

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0$$

finds the extrema

Start Long Derivation

$$S = \int_{x_1}^{x_2} f(Y(x), Y'(x), x)\,dx$$

$$S = \int_{x_1}^{x_2} f\big(y(x) + \alpha \varepsilon(x), y'(x) + \alpha \varepsilon'(x), x\big)\,dx$$

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{d}{d\alpha}\Big[f\big(y(x) + \alpha \varepsilon(x), y'(x) + \alpha \varepsilon'(x), x\big)\Big]\,dx$$

$$= \int_{x_1}^{x_2} \left[\frac{df}{dY}\underbrace{\frac{dY}{d\alpha}} + \frac{df}{dY'}\underbrace{\frac{dY'}{d\alpha}} + \frac{df}{dx}\underbrace{\frac{dx}{d\alpha}}_{0}\right] dx$$

$$\frac{dY}{d\alpha} = \frac{d}{d\alpha}\big(y + \alpha\varepsilon\big) = \varepsilon$$

$$\frac{dY'}{d\alpha} = \frac{d}{d\alpha}\big(y' + \alpha\varepsilon'\big) = \varepsilon'$$

$$\frac{df}{dY} = \frac{df}{dY}\frac{dy}{dy} = \frac{df}{dy}\left[\frac{dy}{dY}\right] \rightarrow 1$$

why? const in y

$$Y(x) = y(x) + \alpha\varepsilon(x)$$

$$\frac{dY}{dy} = 1$$

$$\frac{df}{dY'} = \frac{df}{dy'}$$ same reason

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$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{df}{dy}\varepsilon + \frac{df}{dy'}\varepsilon' \right] dx$$

Set the integral to zero, $\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0$

$$\int_{x_1}^{x_2} \left(\varepsilon \frac{df}{dy} + \varepsilon' \frac{df}{dy'} \right) dx = 0$$

Integrate by parts:

$$\int u'v \, dx = \underbrace{[uv]}_{\text{"surface term" evaluated at } x_1, x_2} - \int uv' \, dx$$

apply to 2nd term

$$\int_{x_1}^{x_2} \varepsilon' \frac{df}{dy'} dx = \underbrace{\left[\varepsilon \frac{df}{dy'} \right]_{x_1}^{x_2}}_{\varepsilon(x_2) = \varepsilon(x_1) = 0} - \int_{x_1}^{x_2} \varepsilon \frac{d}{dx}\left(\frac{df}{dy'} \right) dx$$

surface term vanishes

So,

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(\varepsilon \frac{df}{dy} - \varepsilon \frac{d}{dx}\left(\frac{df}{dy'} \right) \right) dx = 0$$

ok after all that, 6

$$\frac{dS}{d\alpha} = \underbrace{\int_{x_1}^{x_2} \varepsilon(x) \left[\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right)\right] dx}_{} = 0$$

must be true for any $\varepsilon(x)$ so,

$$\boxed{\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0 \quad \text{Euler-Lagrange eqn for 1D}}$$

Return to our line problem

$$l = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

here,

$$f(y, y', x) = \sqrt{1+y'^2}$$

now let's apply the Euler-Lagrange formulation

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$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$\frac{df}{dy} = 0 \qquad \frac{df}{dy'} = \frac{1}{2}\left(1+y'^2\right)^{-1/2}(2y')$$

$$\frac{df}{dy'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$y'(x)$ function of x purely so

$$\boxed{\frac{y'}{\sqrt{1+y'^2}} = \text{constant}}$$

or,

$$y' = c\sqrt{1+y'^2}$$

$$y'^2 = c^2(1+y'^2)$$

$$y'^2(1-c^2) = c^2$$

$$y'^2 = c^2/(1-c^2)$$

So,

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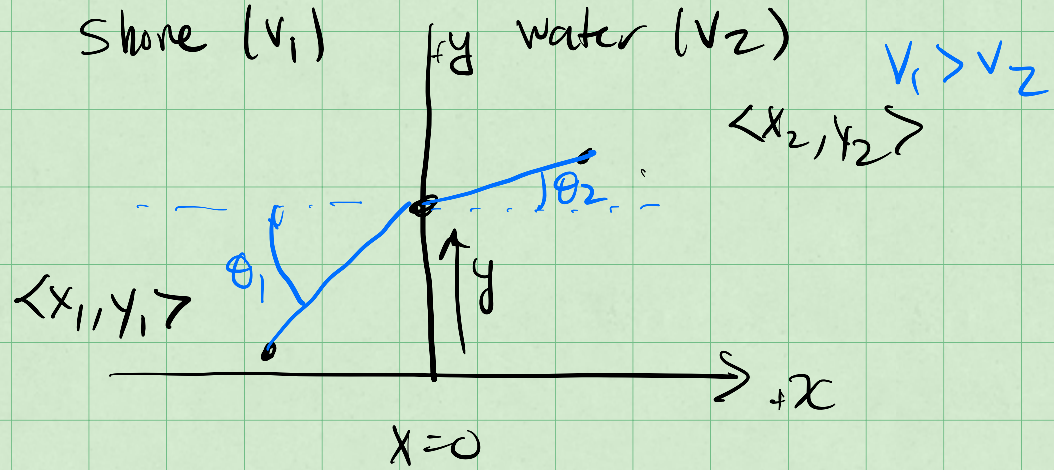
$$y' = \sqrt{\frac{c^2}{1-c^2}} = \text{some other constant} = m$$

$$\frac{dy}{dx} = m = \text{const} \Rightarrow \boxed{y(x) = mx + b}$$

Homework Example: Snell's law

Let's assume you are walking on the beach and need to get to the water quickly. Where do you enter the water to minimize time?



$$T = t_1 + t_2 = \frac{1}{v_1} d_1 + \frac{1}{v_2} d_2$$

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$$d_1 = \sqrt{x_1^2 + (y-y_1)^2} \qquad d_2 = \sqrt{x_2^2 + (y_2-y)^2}$$

$$T = \frac{1}{v_1}\left(x_1^2 + (y-y_1)^2\right)^{1/2} + \frac{1}{v_2}\left(x_2^2 + (y_2-y)^2\right)^{1/2}$$

Find $dT/dy = 0$

$$\frac{dT}{dy} = \frac{1}{2}\frac{1}{v_1}\left(x_1^2 + (y-y_1)^2\right)^{-1/2}(2)(y-y_1)$$

$$+ \frac{1}{2}\frac{1}{v_2}\left(x_2^2 + (y_2-y)^2\right)^{-1/2}(2)(y_2-y)(-1)$$

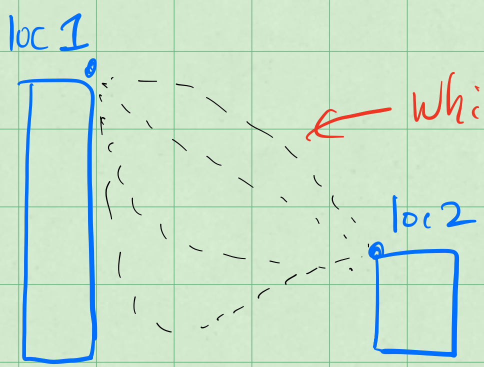
$$= \frac{1}{v_1}\left(\frac{y-y_1}{\sqrt{x_1^2+(y-y_1)^2}}\right) - \frac{1}{v_2}\left(\frac{y_2-y_1}{\sqrt{x_2^2+(y_2-y)^2}}\right) = 0$$

$$= \frac{1}{v_1}\underbrace{\left(\frac{y-y_1}{d_1}\right)}_{\sin\theta_1} - \frac{1}{v_2}\underbrace{\left(\frac{y_2-y_1}{d_2}\right)}_{\sin\theta_2} = 0$$

thus $$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v}$$

An important Example: Brachistrochrone

What shape do we build a frictionless slide to minimize the time to travel between two locations?



which path minimizes travel time?

In 2D we have locations <x, y>

But if we follow the path s then the time it takes to reach 2 from 1 is,

$$\boxed{t = \int_1^2 \frac{ds}{v}}$$

$$t = \int_{loc\,1}^{loc\,2} \frac{ds}{v} \qquad \text{where } ds = \sqrt{dx^2 + dy^2}$$ 11

a generic step in x & y.

What about v?

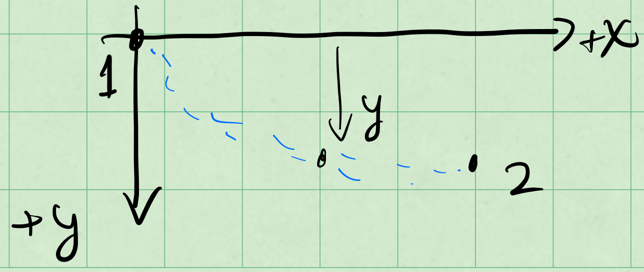
If this is purely gravitational then,

$\Delta K + \Delta U = 0$ so that if $v_i = 0$ then,

$$\frac{1}{2}mv^2 - \frac{1}{2}m(0)^2 + mg(\Delta y) = 0$$

$$v = \sqrt{2g\Delta y}$$ where Δy is the change in height over the fall.

This needs a really coordinate system to make sense.



in this graph then $V = \sqrt{2gy}$ for any location y along the track. (12)

We now write the integral again,

$$t = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}}$$

but,

$$\sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

let $x' = dx/dy$ like before with y'

$$\sqrt{dx^2 + dy^2} = \sqrt{x'^2 + 1}\, dy$$

So that,

$$t = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{(x')^2 + 1}}{\sqrt{y}}\, dy$$

We can identify the function, 13

$$f(x, x', y) = \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}$$

Now we apply the Euler-Lagrange Eqn,

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

Note: $\frac{\partial f}{\partial x} = 0$ so that $f(x, x', y) = f(x', y)$

independent of x.

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0 \quad \text{so} \quad \frac{\partial f}{\partial x'} = \text{constant}$$

$$\frac{\partial f}{\partial x'} = \frac{1}{2} \frac{(x'^2 + 1)^{-1/2}}{\sqrt{y}} (2x')$$

So,

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$$\frac{df}{dx'} = \frac{x'}{\sqrt{y}\sqrt{x'^2+1}} = \underline{\underline{\text{constant}}}$$

square both sides

$$= \sqrt{c}$$

$$\frac{x'^2}{y(x'^2+1)} = c$$

Knowing the solution helps w/ this choice let $c = \frac{1}{2a}$ so,

$$\frac{x'^2}{y(x'^2+1)} = \frac{1}{2a}$$

And Finally,

$$2a\,x'^2 = y(x'^2+1)$$

$$(2a-y)\,x'^2 = y$$

$$x' = \sqrt{\frac{y}{2a-y}}$$

Note:

$$x' = \frac{d\cancel{x}}{dy}$$

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy$$ $\Leftarrow$ this gives the path

x(y) but we need a substitution to solve.

Introduce

$$y = a(1-\cos\theta)$$

$$dy = a\sin\theta$$

$$x(y(\theta)) = \int \sqrt{\frac{y(\theta)}{2a - y(\theta)}}\, dy$$

$$x(\theta) = \int \sqrt{\frac{\cancel{a}(1-\cos\theta)}{2\cancel{a} - \cancel{a}(1-\cos\theta)}}\, a\sin\theta\, d\theta$$

$$= a \int \sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\, \sin\theta\, d\theta$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{(1-\cos\theta)(1+\cos\theta)}$$

$$x(\theta) = a \int \sqrt{\frac{(1-\cos\theta)^2\,(1\cancel{+\cos\theta})}{(1\cancel{+\cos\theta})}}\, d\theta$$

Finally!

$$x(\theta) = a\int(1-\cos\theta)\,d\theta$$

$$x(\theta) = a(\theta - \sin\theta) + \text{Const}$$

if $x = y = 0$ @ $t = 0$ @ $\theta = 0$

$$x(0) = 0 = a(0-0) + \text{const}$$

$$\text{const} = 0$$

Shortest path?

$$x(\theta) = a(\theta - \sin\theta)$$
$$y(\theta) = a(1-\cos\theta)$$

Brachistochrone