Chapter 2

Flows on the Line

2.0 Introduction

In Chapter 1, we introduced the general system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) \end{aligned}$$

and mentioned that its solutions could be visualized as trajectories flowing through an n-dimensional phase space with coordinates $(x_1, \ldots, x_n)$. At the moment, this idea probably strikes you as a mind-bending abstraction. So let's start slowly, beginning here on earth with the simple case $n = 1$. Then we get a single equation of the form

$$\dot{x} = f(x).$$

Here $x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of x. We'll call such equations ***one-dimensional*** or ***first-order systems***.

Before there's any chance of confusion, let's dispense with two fussy points of terminology:

1. The word *system* is being used here in the sense of a dynamical system, not in the classical sense of a collection of two or more equations. Thus a single equation can be a "system."

2. We do not allow f to depend explicitly on time. Time-dependent or "nonautonomous" equations of the form $\dot{x} = f(x, t)$ are more complicated, because one needs *two* pieces of information, x and t, to predict the future state of the system. Thus $\dot{x} = f(x, t)$ should really be regarded as a *two-dimensional* or *second-order* system, and will therefore be discussed later in the book.

2.1 A Geometric Way of Thinking

Pictures are often more helpful than formulas for analyzing nonlinear systems. Here we illustrate this point by a simple example. Along the way we will introduce one of the most basic techniques of dynamics: *interpreting a differential equation as a vector field.*

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x. \tag{1}$$

To emphasize our point about formulas versus pictures, we have chosen one of the few nonlinear equations that can be solved in closed form. We separate the variables and then integrate:

$$dt = \frac{dx}{\sin x},$$

which implies

$$\begin{aligned} t &= \int \csc x \ dx \\ &= -\ln|\csc x + \cot x| + C. \end{aligned}$$

To evaluate the constant C, suppose that $x = x_0$ at $t = 0$. Then $C = \ln|\csc x_0 + \cot x_0|$. Hence the solution is

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|. \tag{2}$$

This result is exact, but a headache to interpret. For example, can you answer the following questions?

1. Suppose $x_0 = \pi/4$. Describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?

2. For an *arbitrary* initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

Think about these questions for a while, to see that formula (2) is not transparent.

In contrast, a graphical analysis of (1) is clear and simple, as shown in Figure 2.1.1. We think of t as time, x as the position of an imaginary particle moving along the real line, and $\dot{x}$ as the velocity of that particle. Then the differential equation $\dot{x} = \sin x$ represents a ***vector field*** on the line: it dictates the velocity vector $\dot{x}$ at each x. To sketch the vector field, it is convenient to plot $\dot{x}$ versus x, and then draw arrows on the x-axis to indicate the corresponding velocity vector at each x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$.

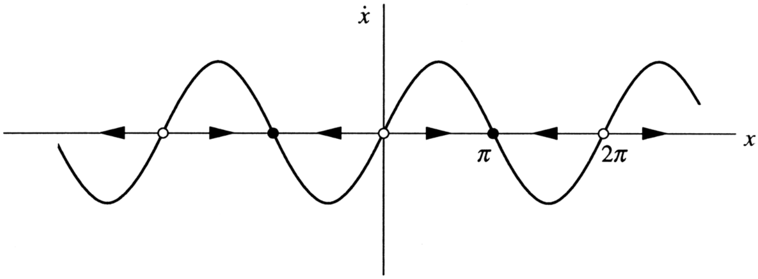

Figure 2.1.1

Here's a more physical way to think about the vector field: imagine that fluid is flowing steadily along the x-axis with a velocity that varies from place to place, according to the rule $\dot{x} = \sin x$. As shown in Figure 2.1.1, the ***flow*** is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At points where $\dot{x} = 0$, there is no flow; such points are therefore called ***fixed points***. You can see that there are two kinds of fixed points in Figure 2.1.1: solid black dots represent ***stable*** fixed points (often called *attractors* or *sinks*, because the flow is toward them) and open circles represent ***unstable*** fixed points (also known as *repellers* or *sources*).

Armed with this picture, we can now easily understand the solutions to the differential equation $\dot{x} = \sin x$. We just start our imaginary particle at x_0 and watch how it is carried along by the flow. This approach allows us to answer the questions above as follows:

1. Figure 2.1.1 shows that a particle starting at $x_0 = \pi/4$ moves to the right faster and faster until it crosses $x = \pi/2$ (where $\sin x$ reaches its maximum). Then the particle starts slowing down and eventually approaches the stable fixed point $x = \pi$ from the left. Thus, the qualitative form of the solution is as shown in Figure 2.1.2.

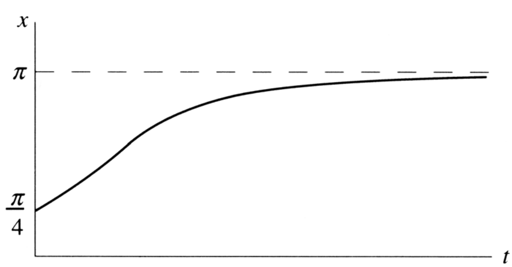

Figure 2.1.2

Note that the curve is concave up at first, and then concave down; this change in concavity corresponds to the initial acceleration for $x < \pi/2$, followed by the deceleration toward $x = \pi$.

2. The same reasoning applies to any initial condition x_0. Figure 2.1.1 shows that if $\dot{x} > 0$ initially, the particle heads to the right and asymptotically approaches the nearest stable fixed point. Similarly, if $\dot{x} < 0$ initially, the particle approaches the nearest stable fixed point to its left. If $\dot{x} = 0$, then x remains constant. The qualitative form of the solution for any initial condition is sketched in Figure 2.1.3.

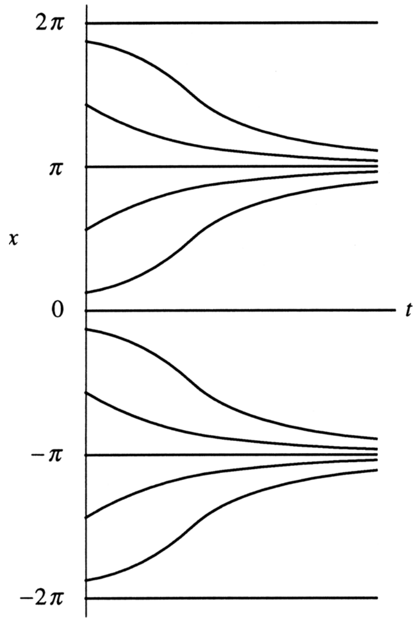

Figure 2.1.3

In all honesty, we should admit that a picture can't tell us certain *quantitative* things: for instance, we don't know the time at which the speed $|\dot{x}|$ is greatest. But in many cases *qualitative* information is what we care about, and then pictures are fine.

2.2 Fixed Points and Stability

The ideas developed in the last section can be extended to any one-dimensional system $\dot{x} = f(x)$. We just need to draw the graph of $f(x)$ and then use it to sketch the vector field on the real line (the x-axis in Figure 2.2.1).

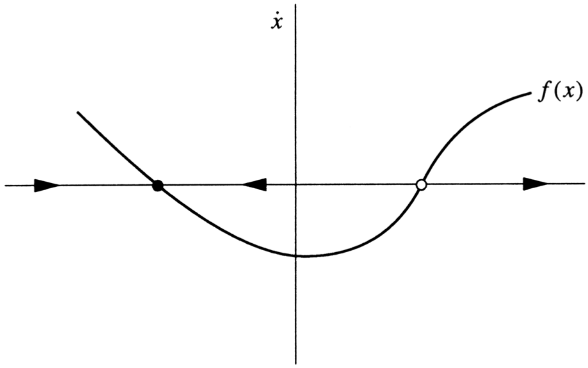

Figure 2.2.1

As before, we imagine that a fluid is flowing along the real line with a local velocity $f(x)$. This imaginary fluid is called the phase fluid, and the real line is the phase space. The flow is to the right where $f(x) > 0$ and to the left where $f(x) < 0$. To find the solution to $\dot{x} = f(x)$ starting from an arbitrary initial condition x_0, we place an imaginary particle (known as a ***phase point***) at x_0 and watch how it is carried along by the flow. As time goes on, the phase point moves along the x-axis according to some function $x(t)$. This function is called the ***trajectory*** based at x_0, and it represents the solution of the differential equation starting from the initial condition x_0. A picture like Figure 2.2.1, which shows all the qualitatively different trajectories of the system, is called a ***phase portrait***.

The appearance of the phase portrait is controlled by the fixed points x^*, defined by $f(x^*) = 0$; they correspond to stagnation points of the flow. In Figure 2.2.1, the solid black dot is a stable fixed point (the local flow is toward it) and the open dot is an unstable fixed point (the flow is away from it).

In terms of the original differential equation, fixed points represent ***equilibrium*** solutions (sometimes called steady, constant, or rest solutions, since if $x = x^*$ initially, then $x(t) = x^*$ for all time). An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time. Thus stable equilibria are represented geometrically by stable fixed points. Conversely, unstable equilibria, in which disturbances grow in time, are represented by unstable fixed points.

Example 2.2.1:

Find all the fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

Solution: Here $f(x) = x^2 - 1$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^*. Thus $x^* = \pm 1$. To determine stability, we plot $x^2 - 1$ and then sketch the vector field (Figure 2.2.2). The flow is to the right where $x^2 - 1 > 0$

and to the left where $x^2 - 1 < 0$. Thus $x^* = -1$ is stable, and $x^* = 1$ is unstable. ■

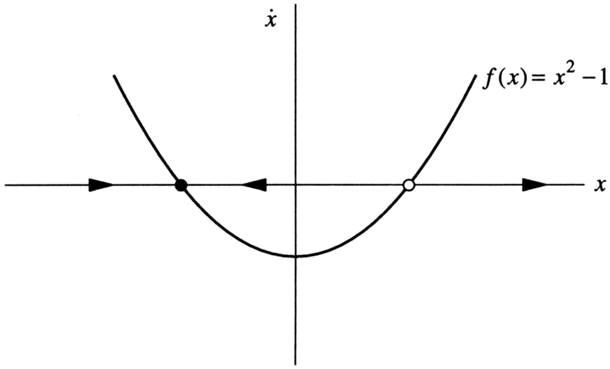

Figure 2.2.2

Note that the definition of stable equilibrium is based on *small* disturbances; certain large disturbances may fail to decay. In Example 2.2.1, all small disturbances to $x^* = -1$ will decay, but a large disturbance that sends x to the right of $x = 1$ will *not* decay—in fact, the phase point will be repelled out to $+\infty$. To emphasize this aspect of stability, we sometimes say that $x^* = -1$ is *locally stable*, but not globally stable.

Example 2.2.2:

Consider the electrical circuit shown in Figure 2.2.3.

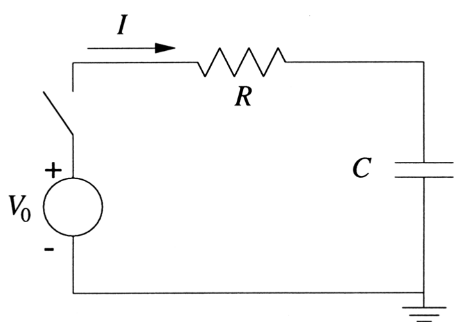

Figure 2.2.3

A resistor R and a capacitor C are in series with a battery of constant dc voltage V_0. Suppose that the switch is closed at $t = 0$, and that there is no charge on the capacitor initially. Let $Q(t)$ denote the charge on the capacitor at time $t \geq 0$. Sketch the graph of $Q(t)$.

Solution: This type of circuit problem is probably familiar to you. It is governed by linear equations and can be solved analytically, but we prefer to illustrate the geometric approach.

First we write the circuit equations. As we go around the circuit, the total voltage drop must equal zero; hence $-V_0 + RI + Q/C = 0$, where I is the current flowing through the resistor. This current causes charge to accumulate on the capacitor at a rate $\dot{Q} = I$. Hence

$$-V_0 + R\dot{Q} + Q/C = 0$$

which gives

$$\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}.$$

The graph of $f(Q)$ is a straight line with a negative slope (Figure 2.2.4).

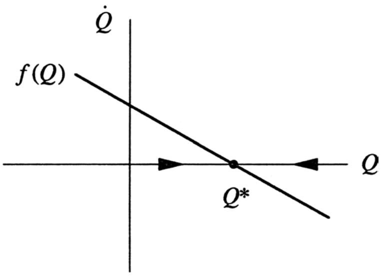

Figure 2.2.4

The corresponding vector field has a fixed point where $f(Q) = 0$, which occurs at $Q^* = CV_0$. The flow is to the right where $f(Q) > 0$ and to the left where $f(Q) < 0$. Thus the flow is always toward Q^*; it is a *stable* fixed point. In fact, it is ***globally stable***, in the sense that it is approached from *all* initial conditions.

To sketch $Q(t)$, we start a phase point at the origin of Figure 2.2.4 and imagine how it would move. The flow carries the phase point monotonically toward Q^*. Its speed $\dot{Q}$ decreases linearly as it approaches the fixed point; therefore $Q(t)$ is increasing and concave down, as shown in Figure 2.2.5. ■

Example 2.2.3:

Sketch the phase portrait corresponding to $\dot{x} = x - \cos x$, and determine the stability of all the fixed points.

Solution: One approach would be to plot the function $f(x) = x - \cos x$ and then sketch the associated vector field. This method is valid, but it requires you to figure out what the graph of $x - \cos x$ looks like.

There's an easier solution, which exploits the fact that we know how to graph $y = x$ and $y = \cos x$ *separately.* We plot both graphs on the same axes and then observe that they intersect in exactly one point (Figure 2.2.6).

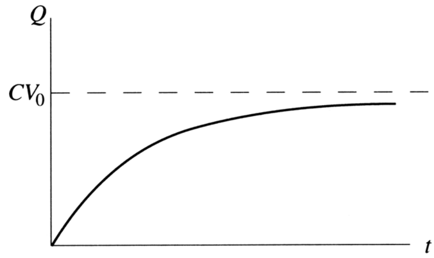

Figure 2.2.5

This intersection corresponds to a fixed point, since $x^* = \cos x^*$ and therefore $f(x^*) = 0$. Moreover, when the line lies above the cosine curve, we have $x > \cos x$ and so $\dot{x} > 0$: the flow is to the right. Similarly, the flow is to the left where the line is below the cosine curve.

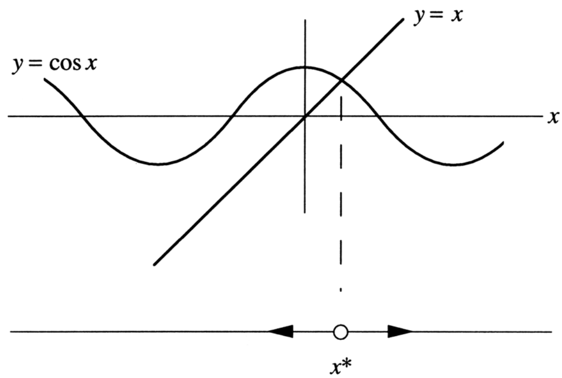

Figure 2.2.6

Hence x^* is the only fixed point, and it is unstable. Note that we can classify the stability of x^*, even though we don't have a formula for x^* itself! ■

2.3 Population Growth

The simplest model for the growth of a population of organisms is $\dot{N} = rN$, where $N(t)$ is the population at time t, and $r > 0$ is the growth rate. This model predicts exponential growth: $N(t) = N_0 e^{rt}$, where N_0 is the population at $t = 0$.

Of course such exponential growth cannot go on forever. To model the effects of overcrowding and limited resources, population biologists and demographers often assume that the per capita growth rate $\dot{N}/N$ decreases when N becomes sufficiently large, as shown in Figure 2.3.1. For small N, the growth rate equals r, just as before. However, for populations larger than a certain ***carrying capacity*** K, the growth rate actually becomes negative; the death rate is higher than the birth rate.

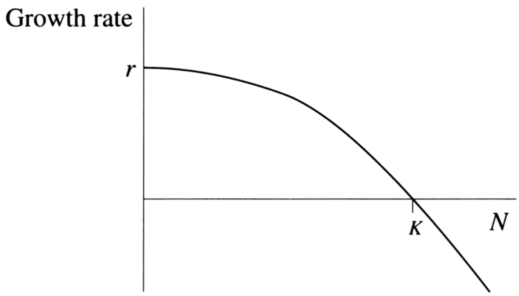

Figure 2.3.1

A mathematically convenient way to incorporate these ideas is to assume that the per capita growth rate $\dot{N}/N$ decreases *linearly* with N (Figure 2.3.2).

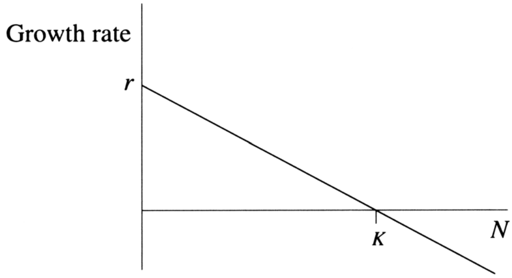

Figure 2.3.2

This leads to the ***logistic equation***

$$\dot{N} = rN\left(1 - \frac{N}{K}\right),$$

first suggested to describe the growth of human populations by Verhulst in 1838. This equation can be solved analytically (Exercise 2.3.1) but once again we prefer a graphical approach. We plot $\dot{N}$ versus N to see what the vector field looks like. Note that we plot only $N \geq 0$, since it makes no sense to think about a negative population (Figure 2.3.3).

Fixed points occur at $N^* = 0$ and $N^* = K$, as found by setting $\dot{N} = 0$ and solving for N. By looking at the flow in Figure 2.3.3, we see that $N^* = 0$ is an unstable fixed point and $N^* = K$ is a stable fixed point. In biological terms,

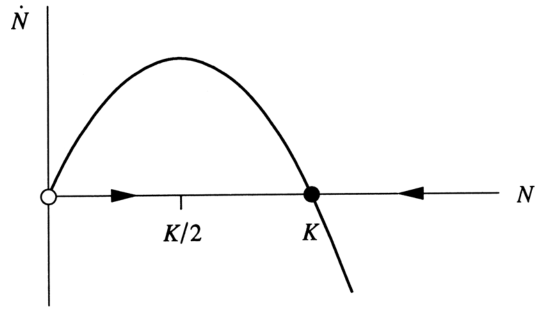

Figure 2.3.3

$N = 0$ is an unstable equilibrium: a small population will grow exponentially fast and run away from $N = 0$. On the other hand, if N is disturbed slightly from K, the disturbance will decay monotonically and $N(t) \to K$ as $t \to \infty$.

In fact, Figure 2.3.3 shows that if we start a phase point at *any* $N_0 > 0$, it will always flow toward $N = K$. Hence *the population always approaches the carrying capacity.*

The only exception is if $N_0 = 0$; then there's nobody around to start reproducing, and so $N = 0$ for all time. (The model does not allow for spontaneous generation!)

Figure 2.3.3 also allows us to deduce the qualitative shape of the solutions. For example, if $N_0 < K/2$, the phase point moves faster and faster until it crosses $N = K/2$, where the parabola in Figure 2.3.3 reaches its maximum. Then the phase point slows down and eventually creeps toward $N = K$. In biological terms, this means that the population initially grows in an accelerating fashion, and the graph of $N(t)$ is concave up. But after $N = K/2$, the derivative $\dot{N}$ begins to decrease, and so $N(t)$ is concave down as it asymptotes to the horizontal line $N = K$ (Figure 2.3.4). Thus the graph of $N(t)$ is S-shaped or *sigmoid* for $N_0 < K/2$.

Something qualitatively different occurs if the initial condition N_0 lies between $K/2$ and K; now the solutions are decelerating from the start. Hence these solutions are concave down for all t. If the population initially exceeds the carrying capacity ($N_0 > K$), then $N(t)$ decreases toward $N = K$ and is concave up. Finally, if $N_0 = 0$ or $N_0 = K$, then the population stays constant.

Critique of the Logistic Model

Before leaving this example, we should make a few comments about the biological validity of the logistic equation. The algebraic form of the model is not to be taken literally. The model should really be regarded as a metaphor for populations that have a tendency to grow from zero population up to some carrying capacity K.

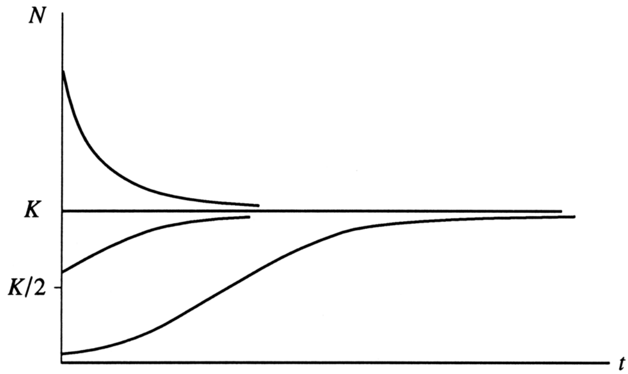

Figure 2.3.4

Originally a much stricter interpretation was proposed, and the model was argued to be a universal law of growth (Pearl 1927). The logistic equation was tested in laboratory experiments in which colonies of bacteria, yeast, or other simple organisms were grown in conditions of constant climate, food supply, and absence of predators. For a good review of this literature, see Krebs (1972, pp. 190–200). These experiments often yielded sigmoid growth curves, in some cases with an impressive match to the logistic predictions.

On the other hand, the agreement was much worse for fruit flies, flour beetles, and other organisms that have complex life cycles involving eggs, larvae, pupae, and adults. In these organisms, the predicted asymptotic approach to a steady carrying capacity was never observed—instead the populations exhibited large, persistent fluctuations after an initial period of logistic growth. See Krebs (1972) for a discussion of the possible causes of these fluctuations, including age structure and time-delayed effects of overcrowding in the population.

For further reading on population biology, see Pielou (1969) or May (1981). Edelstein–Keshet (1988) and Murray (2002, 2003) are excellent textbooks on mathematical biology in general.

2.4 Linear Stability Analysis

So far we have relied on graphical methods to determine the stability of fixed points. Frequently one would like to have a more quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by *linearizing* about a fixed point, as we now explain.