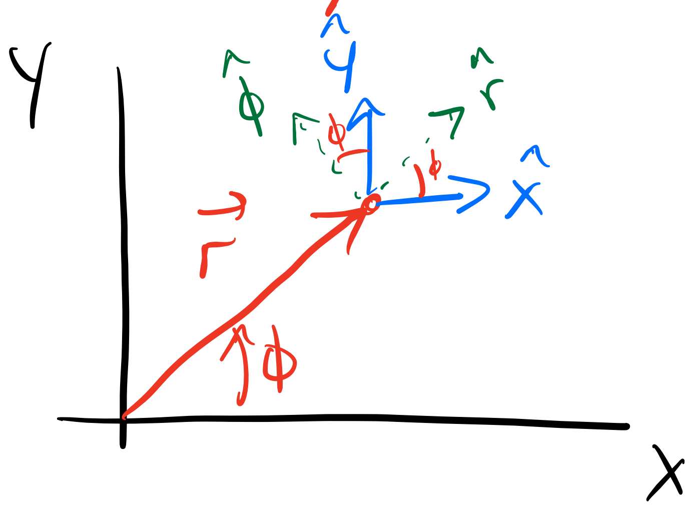
Deriving Newton's 2nd Law in Plane Polar

1. Draw $\vec{r}$
2. compute $d\vec{r}/dt = \vec{v}$
3. compute $d^2\vec{r}/dt^2 = \vec{a}$
4. investigate $\vec{F} = m\vec{a}$

① 

$\vec{r} = r\hat{r}$ where $r = |\vec{r}|$

② $$\frac{d\vec{r}}{dt} = \underbrace{\frac{dr}{dt}}_{\dot{r}}\hat{r} + r\underbrace{\frac{d\hat{r}}{dt}}_{?}$$

$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

do in cartesian

$$\hookrightarrow \frac{d\hat{r}}{dt} = -\dot{\phi}\sin\phi\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y} = \dot{\phi}\hat{\phi}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

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$$\vec{a} = \frac{d\vec{v}}{dt} = \underbrace{\frac{d\dot{r}}{dt}}_{\ddot{r}}\hat{r} + \dot{r}\underbrace{\frac{d\hat{r}}{dt}}_{\dot{\phi}\hat{\phi}} + \underbrace{\frac{dr}{dt}}_{\dot{r}}\dot{\phi}\hat{\phi} + r\underbrace{\frac{d\dot{\phi}}{dt}}_{\ddot{\phi}}\hat{\phi} + r\dot{\phi}\underbrace{\frac{d\hat{\phi}}{dt}}_{?}$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\cos\phi\,\hat{x} - \dot{\phi}\sin\phi\,\hat{y} = -\dot{\phi}\hat{r}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\left(-\dot{\phi}\hat{r}\right)$$

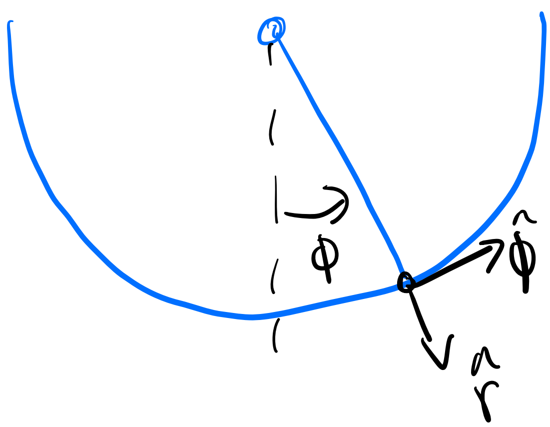
$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

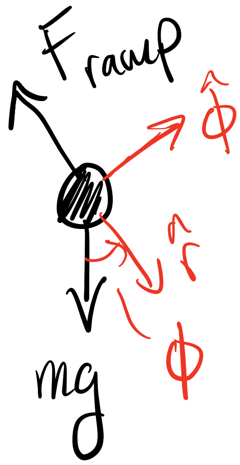
4

$$\vec{F}_{net} = m\vec{a} \Rightarrow \vec{F}_r + \vec{F}_\phi = m\left(\vec{a}_r + \vec{a}_\phi\right)$$

$$F_r = m\left(\ddot{r} - r\dot{\phi}^2\right) \qquad F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)$$

Skateboard Example





$$\sum F_r = ma_r = -F_{ramp} + mg\cos\phi = m(\ddot{r} - r\dot{\phi}^2)$$

Note $r = R$ so $\ddot{r} = 0$,

$$-F_{ramp} + mg\cos\phi = -mR\dot{\phi}^2$$

$$\sum F_\phi = ma_\phi = -mg\sin\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

Note: $r = R$ $\dot{r} = 0$

$$-mg\sin\phi = mR\ddot{\phi}$$

or $$\boxed{\ddot{\phi} = -\frac{g}{R}\sin\phi}$$

Assume small osc.

$\sin\phi \approx \phi$

$$\ddot{\phi} = -\frac{g}{R}\phi \implies \ddot{x} = -\omega^2 x \,?$$

$$\omega^2 = g/R \quad \text{so} \quad \omega = \sqrt{g/R}$$

$$\phi(t) = A\cos(\omega t) + B\sin(\omega t)$$ general soln.

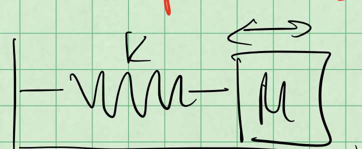
Lagrangian Dynamics

So far you have studied much of physics from the perspective of Netwonian Mechanics. The idea behind that is

1. we can identify all the interactions with a body (ie. the forces + torques)
2. we can write models of those interactions using mathematical functions (i.e. $F=-kx$, βv^2)
3. We vectorially add all the individual interactions to find the net result (i.e. $\sum \vec{F}_i = \vec{F}_{net}$)
4. We apply Newton's Second to the body $m\ddot{\vec{r}} = \vec{F}_{net}$

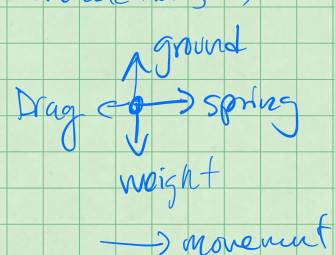
This is our equation of motion (EOM)

Quick Example: 1D SHO w/ linear drag



β drag coeff.

1. Interactions?



2. $-$ground $=$ weight $= mg$ $(\hat{y})$
spring $= -kx$ Drag $= -\beta\dot{x}$ $(\hat{x})$

(3) $\sum \vec{F}_i$ $\rightarrow$ weight + ground $= 0$ nothing interesting

$\rightarrow$ spring + drag $= F_{net,x}$

(2)

$$F_{net,x} = -kx - \beta\dot{x}$$

(4) $$m\ddot{x} = -kx - \beta\dot{x} \Rightarrow \boxed{m\ddot{x} + \beta\dot{x} + kx = 0}$$

What about Lagrange?

The Lagrange Formulation is rooted in classical optimization, but it is equivalent to Newton's work. However, it uses energy (a scalar) to do so. This means we can exploit coordinate transformations that can't change the scalar value (of energy).

Epistomelogically speaking, Lagrange's approach is an exploration of phase space to determine paths the dynamical system can take. You can think of this as one level above plotting the known phase space, b/c we don't yet know the EOMs.

Let's do an example and then backup.

Lagrange SHO

As you will learn to get the EOM for a non dissipative system we form a function called the Lagrangian,

$$L = T - U \qquad \text{where} \quad \begin{array}{l} T = \text{kinetic energy} \\ U = \text{potential energy} \end{array}$$

$$\boxed{T = +\tfrac{1}{2}m\dot{x}^2 \qquad U = +\tfrac{1}{2}kx^2 \qquad \text{1D SHO}}$$

Now we follow the optimization routine,

$$\boxed{\frac{dL}{dx} - \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = 0 \qquad \text{1D Euler-Lagrange Eqn}}$$

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2 \qquad \frac{dL}{dx} = -kx \qquad \frac{dL}{d\dot{x}} = m\dot{x}$$

thus,

$$-kx - \frac{d}{dt}(m\dot{x}) = -kx - m\ddot{x} = 0$$

so, $m\ddot{x} + kx = 0$ or $\boxed{\ddot{x} = -\frac{k}{m}x}$

MAGIC?

<u>no</u> <u>just</u> <u>framing</u> <u>mechanics</u> <u>differently</u>

Calculus of Variations

This is not magic; it's an application of an approach that conceives of letting a system evolve so long as it starts from given location in phase space and moves to another given location.

+ We find the "optimal" path.

⇒ This is called "variational calculus" or "calculus of variations"

* A discussion of the origins and base theory are not here. But references are included (Boas, Weber & Arfken, Goldstein all cover that).

Statement of the Calc of Variation Problem

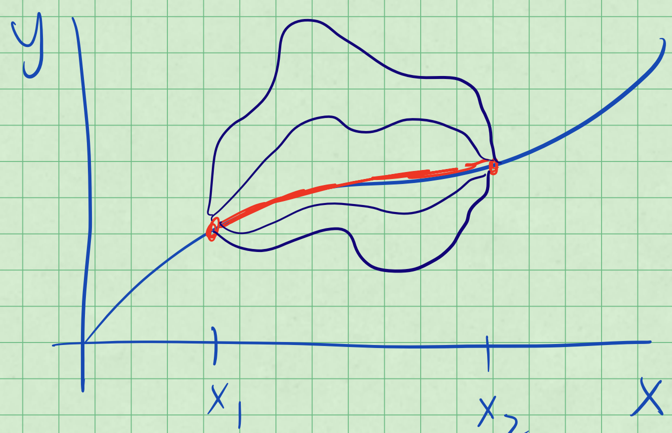
Consider an integral J of the form,

$$J = \int_{x_1}^{x_2} f(y, y')\, dx \qquad \text{where } y = y(x), \quad y' = \frac{dy(x)}{dx}$$

We aim to determine how we can choose f so that J is an extremum (i.e. a min or max)
typically a min in mechanics

Thought Experiment

Assume we know the path through phase space



true path (an extrema of J)

alternative paths with larger J's.

$y(\alpha, x) = y(0, x)$ where $\alpha = 0$ represents the known path

$y(\alpha, x) = y(0, x) + \alpha\eta(x)$ where $\eta(x)$ is additional function describing possible alt. paths.

note: $\eta(x_1) = 0$

$\eta(x_2) = 0$

Finding the extrema means taking $\left.\frac{dJ}{d\alpha}\right|_{\alpha=0}$ and setting that to zero.

$$J(\alpha) = \int_{x_1}^{x_2} f(y(\alpha, x), y'(\alpha, x))\,dx$$

I won't derive in detail. It's in most books that taking $dJ/d\alpha\big|_{\alpha=0} = 0$ gives

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{Euler equations}}$$

Example: Brachistrochrone (in class)

Hamilton's Principle

The path followed by a dynamical system minimizes the time integral of $T-U$.

Mathematically,

$$\delta\int_{t_1}^{t_2}(T-U)\,dt = 0$$

with $T = T(\dot{x}_i)$ $\quad U = U(x_i)$

$$L \equiv T - U = L(x_i, \dot{x}_i)$$

So

$$\delta\int_{t_1}^{t_2} L(x_i, \dot{x}_i)\,dt = 0$$

such that

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \quad i = 1, 2, 3,}$$

We have Lagrange's EOM! for cartesian $\underline{x}_i$ 7

$$\boxed{\frac{dL}{dx_i} - \frac{d}{dt}\left(\frac{dL}{d\dot{x}_i}\right) = 0 \qquad i = 1, 2, 3 \quad \boxed{x, y, z} \text{ think}}$$

Generalized Coordinates

As it turns out, Lagrangian Dynamics can make use of any phase coordinate $(\theta, \phi, x, y, \rho, z, \text{etc.})$ and it's first derivative $(\dot{\theta}, \dot{\phi}, \dot{x}, \dot{y}, \dot{\rho}, \dot{z}, \text{etc})$

So In fact for any set of general coordinates,

$$\vec{q} = \langle q_1, q_2, q_3, \dots, q_N \rangle$$

and their derivatives,

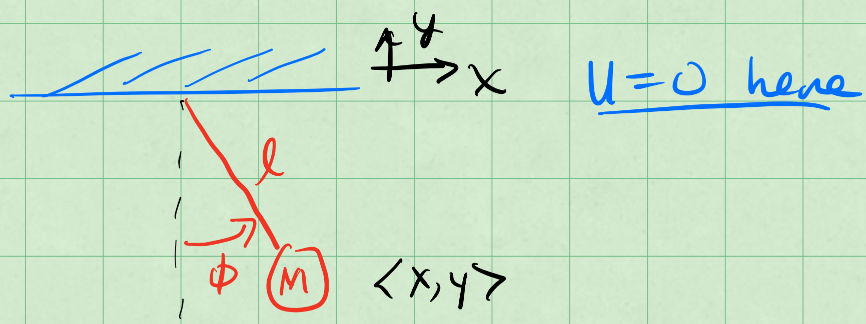
$$\dot{\vec{q}} = \langle \dot{q}_1, \dot{q}_2, \dot{q}_3, \dots, \dot{q}_N \rangle$$

give rise to a general set of N equations (subject to reductions due to Constraint eqns. (more later))

$$\boxed{\frac{dL}{dq_i} - \frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) = 0 \qquad i = 1, 2, \dots, N}$$

1

To get some intuition about Lagrangians lets look at a problem that we know well: The Plane Pendulum



Here the pendulum swings in the x-y plane. Let's use x, y coordinates to "naively" setup the Lagrangian and see what happens.

$$T(\dot{x},\dot{y}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$U(x,y) = U(y) = +mgy$$

so we will get Two EOMs:

1 $\frac{dL}{dx} - \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = 0$ 2 $\frac{dL}{dy} - \frac{d}{dt}\left(\frac{dL}{d\dot{y}}\right) = 0$

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$ 2

1

$$\underbrace{0 - \frac{d}{dt}(m\dot{x})}_{p_x \text{ conserved?}} = 0 \qquad P_x = m\dot{x}$$

← that can't be right

2

$$-mg - \frac{d}{dt}(m\dot{y}) = 0 \Rightarrow \underbrace{\ddot{y} = -g}_{\text{no tension force?}}$$

We failed to include the constraint!

the pendulum length is fixed!

$$\boxed{x^2 + y^2 = \ell^2} \quad \text{so,} \quad \boxed{\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2 = \ell^2\dot{\phi}^2}$$

Constraint ⟶ implication

So as you might have guessed (r, ϕ) was a better choice of coords.

$$\mathcal{L}(x, y, \dot{x}, \dot{y}, t) = \mathcal{L}(r, \phi, \dot{r}, \dot{\phi}, t)$$

3

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\ell^2\dot{\phi}^2 = T(\dot{r},\dot{\phi})$$

$$U = mgy = -mg\ell\cos\phi = U(r,\phi)$$

$$L = T - U = \frac{1}{2}m\ell^2\dot{\phi}^2 + mg\ell\cos\phi$$

$\mathcal{L}(\phi,\dot{\phi})$! no $r, \dot{r}$ 1D EOM!

$$\frac{dL}{d\phi} - \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right) = 0$$

$$-mg\ell\sin\phi - \frac{d}{dt}(m\ell^2\dot{\phi}) = 0$$

ϕ EOM:

$$m\ell^2\ddot{\phi} = -mg\ell\sin\phi$$

$$\boxed{\ddot{\phi} = \frac{-g}{\ell}\sin\phi}$$

which for small ϕ,

$\sin\phi \simeq \phi$,

$$\ddot{\phi} = -\frac{g}{l}\phi = -\omega^2\phi$$

$$\phi(t) = A\cos(\omega t) + B\sin(\omega t)$$

with $\omega^2 = g/l$!

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