Driving the Oscillator

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We've seen that we can get different kinds of motion with different damping of the SHO.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Here the relative strengths of β^2 & ω_0^2 affect the type of motion we observed.

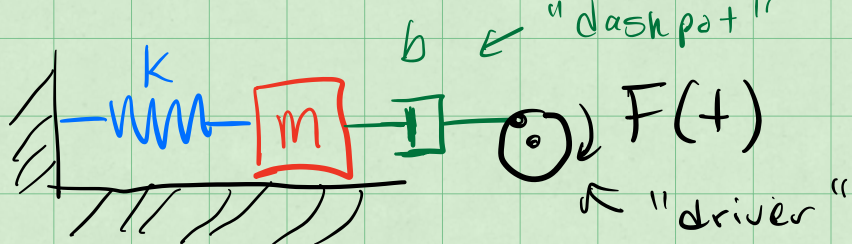
But the motion always decays because the damping removes energy from the system.

What if we kept putting energy in?

$\Rightarrow$ Enter Driven Oscillations

(sometimes called "Forced" osc.)

Here we add time dependent driver,



In terms of the quantities above, this differential equation is readily written,

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Denoting $F(t)/m = f(t)$, driving per unit mass

$$\ddot{x} + 2\beta x + \omega_0^2 x = f(t)$$

1D driven oscillator

With $\omega_0^2 = k/m$ & $2\beta = b/m$

Note: this differential equation remains linear

Why? Because the operations on $x(t)$ are linear

Linearity of Differential Equations

A linear differential eqn has many nice properties $\rightarrow$ in particular, superposition of solutions & uniqueness of solutions.

So it is worth knowing when you have one. 3

$\frac{d}{dt}$ is a linear operation. Why?

let $x(t) = x_1(t) + x_2(t)$

$$\frac{d}{dt}\left(x(t)\right) = \frac{d}{dt}\left(x_1(t) + x_2(t)\right)$$

$$= \frac{dx_1}{dt} + \frac{dx_2}{dt}$$

Because we can distribute the operation, $\frac{d}{dt}$ is a Linear Differential Operator

Thus constants and simple derivatives are linear operators. So are their linear sums.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \underbrace{\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)}_{D}x$$

D, linear differential operator

So, $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$

can be written: $Dx = f$

Effect on Solutions?

B/C D is a linear operator,

(1) $D(ax) = a\,Dx$

(2) $D(x_1 + x_2) = Dx_1 + Dx_2$

(3) $D(ax_1 + bx_2) = a\,Dx_1 + b\,Dx_2$

For $Dx = f$, we have to be careful.

Assume there's a solution to the homogeneous differential Eqn,

$$Dx_h = 0$$

where $x_h(t)$ is the homogeneous solution, then this solution can be added to any solution to

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$$Dx = f$$

without affecting the solution to

Why?

$D(x_p + x_h) = f$ where x_p is the solution to

$D x_p + \underbrace{D x_h}_{0} = f \leftarrow D x_p = f$

b/c homogeneous solution!

$D x_p = f$ ✓

x_p is called the "particular" solution.

Summary: When solving $Dx = f$, the solution is $x = x_p + x_h$ where

$D x_h = 0$ as before

thus, $x_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ which die out as $t \to \infty$

Moreover, $x_h(t)$ satisfies two unknowns set by initial conditions. So x_p will not depend on ICs. However, it is particular to the driver, $f(t)$. 6

Sinusoidal Driving

Let $f(t) = f_0 \cos(\omega t)$ Note $\omega \neq \omega_0$

so that,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

if $f(t) = f_0 \sin(\omega_0 t)$ then,

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$ is the valid Diffy Q.

We cleverly combine them with a linear operation

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let $z(t) = x(t) + iy(t)$

then $f(t) = f_0\left(\cos(\omega t) + i\sin(\omega t)\right)$

$$f(t) = f_0 e^{i\omega t}$$

Thus we seek the particular soln. to,

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

Note C is undetermined but it must not be set by initial conditions.

try $z(t) = Ce^{i\omega t}$

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

$$-\omega^2 Ce^{i\omega t} + 2i\beta\omega Ce^{i\omega t} + \omega_0^2 Ce^{i\omega t} = f_0 e^{i\omega t}$$

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)\, C\cancel{e^{i\omega t}} = f_0 \cancel{e^{i\omega t}}$$

$$\boxed{C = \frac{f_0}{(\omega_0^2 - \omega^2 + 2i\beta\omega)}}$$

fully determined from setup

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We found C and it is fully determined. But it's complex. Ultimately we will want a real valued solution,

$$Re(z) = x(t) \quad Im(z) = y(t)$$

Remember, we can write C as a magnitude, A, and phase, δ.

$$C = Ae^{-i\delta}$$

$$A^2 = CC^*$$

$$A^2 = \left(\frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}\right)\left(\frac{f_0}{\omega_0^2 - \omega^2 - 2i\beta\omega}\right)$$

$$\boxed{A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

What about the phase, δ?

$Ce^{-i\delta}$ gives

$$\underset{\substack{\uparrow \\ \text{real}}}{f_0}\, e^{i\delta} = \underset{\substack{\uparrow \\ \text{real}}}{A}\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right)$$

$e^{i\delta}$ has the same phase as $(\omega_0^2 - \omega^2) + i\,2\beta\omega$

or,

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$$\boxed{\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)}$$

phase fully determined

We proposed a solution $z(t) = Ce^{i\omega t}$ 11

which became $z(t) = Ae^{-i\delta} e^{i\omega t}$

$$z(t) = \underset{\nwarrow \text{ real}}{A} e^{i(\omega t - \delta)}$$

$x_p(t) = A\cos(\omega t - \delta)$ particular soln.

And finally,

$$x(t) = x_p(t) + x_h(t)$$

$$x(t) = \underbrace{A\cos(\omega t - \delta)}_{\text{long term behavior}} + \underbrace{C_1 e^{r_1 t} + C_2 e^{r_2 t}}_{\text{"transient solutions" decay as } t \to \infty}$$

For weakly damped $\beta^2 < \omega_0^2$ (typical case)

$$\boxed{\begin{gathered} x(t) = A\cos(\omega t - \delta) + A_{tr} e^{-\beta t} \cos(\omega_1 t - \delta_{tr}) \\ \text{where } \omega_1 = \sqrt{\omega_0^2 - \beta^2} \text{ per usual} \end{gathered}}$$

Resonance & Tuning

A curious thing about long term behavior is that we can observe large amplitudes at particular choices of driving.

$$x_{\text{longterm}}(t) = A\cos(\omega t - \delta)$$

where,

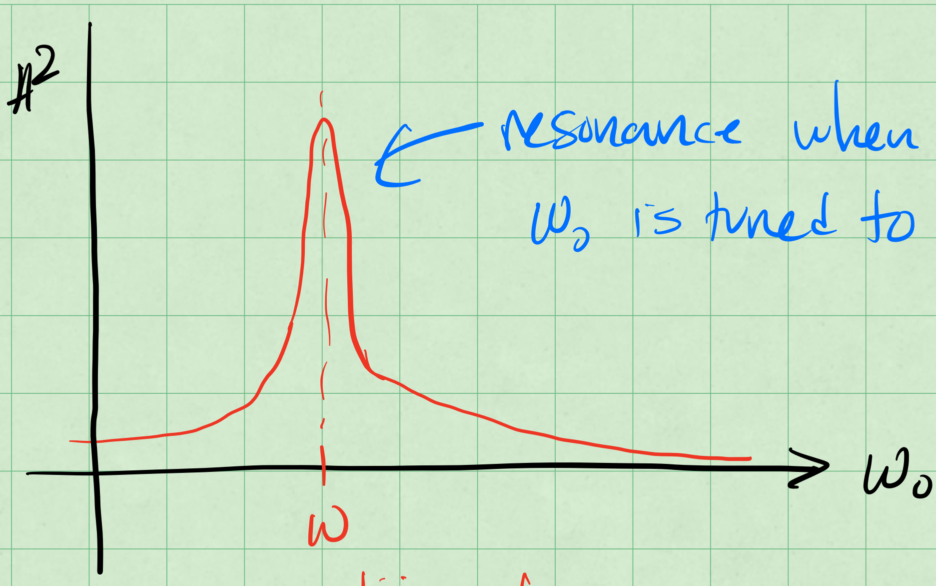
$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

let β be small so that $4\beta^2\omega^2$ is small

if ω_0^2 and ω^2 are far apart, denom is large, thus A^2 small!

if ω_0^2 and ω^2 are close, denom is small, thus A^2 large!

↳ resonant behavior.



resonance when ω_0 is tuned to ω.

let K/m change

driver freq.

Let's focus on the denominator

$$\boxed{(\omega_0^2-\omega^2)^2+4\beta\omega^2}$$

Amplitude max when a minimum.

this depends a bit on which thing you change. the oscillator (ω_0) or the driver (ω)

Case 1: Tune ω_0 to ω (Radio)

obviously when $\omega_0=\omega$ then

denom is $4\beta^2\omega_0^2$

$$A = f_0/2\beta\omega_0$$

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Case 2: Tune ω to fixed ω_0 (adjust driver)

$$\frac{d}{d\omega}\left((\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right)=0$$

$$2(\omega_0^2-\omega^2)(-2\omega)+8\beta^2\omega=0$$

$$-4\omega_0^2\omega+4\omega^3+8\beta^2\omega=0$$

$$4\omega(\omega^2-\omega_0^2+2\beta^2)=0$$

$\underbrace{\omega=0}_{\text{no driver}}$ $\qquad$ $\omega^2=\omega_0^2-2\beta^2$

$$\boxed{\omega_2=\sqrt{\omega_0^2-2\beta^2}}$$

max response when $\omega_0^2 \gg \beta^2$

$\underline{\underline{\omega_2 \simeq \omega_0}}$