CW 6 - Phase Space, Nonlinear Dynamics

Thus far we have developed an approach to solving for particular trajectories of different classical mechanics systems.

Typically, we work to develop the differential equation that describes how the system evolves → the equation of motion.

usually of the form,

$$\ddot{\vec{x}} = f(\vec{x}, \dot{\vec{x}}, t)$$

generic equation of motion for $\vec{x}$.

or in 1D,

$$\ddot{x} = g(x, \dot{x}, t)$$

The approach has been to find ways to solve these equations for given initial conditions (x_0, v_0 @ t_0) to get trajectories

$x(t)$ & $v(t)$ for $t > t_0$

- However, some equations of motion (in fact, most) cannot be solved in "closed form", which is what gives rise to $x(t)$ & $v(t)$.
- Moreover, it seems pretty inefficient to solve for individual trajectories to understand what the system is doing.
- Typically, we care more about the potential solutions or qualitatively different solutions

Example: Nonlinear 1st Order ODE

let $\dot{x} = \sin x$ Find $x(t)$

$$\frac{dx}{dt} = \sin x \xRightarrow{\text{sep of var}} \frac{dx}{\sin x} = dt$$

$$\int_0^t dt' = \int_{x_0}^{x(t)} \frac{dx'}{\sin x'}$$

$$\int \frac{dx}{\sin x} = \ln\left(|\csc x - \cot x|\right)$$

$$t = \ln\left(\left|\csc x(t) - \cos x(t)\right|\right) - \ln\left(\left|\csc x_0 - \cot x_0\right|\right)$$

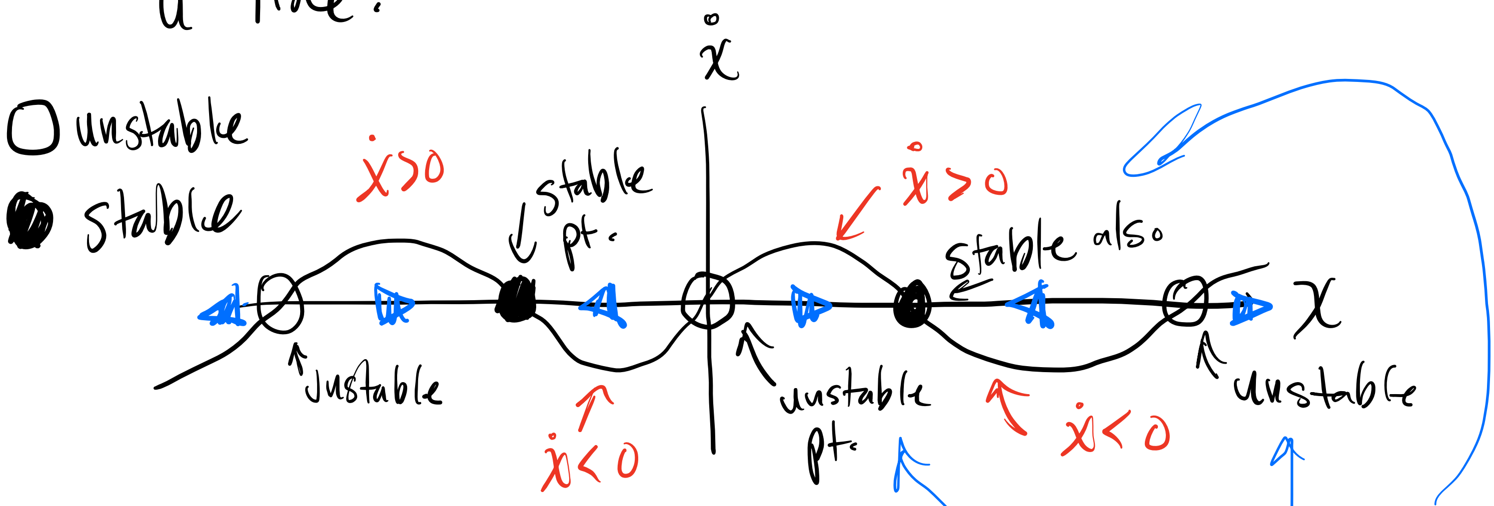
OOF. $x(t) = ?$ More importantly, what does the motion look like?

We need a new tool set to deal w/ this

Let's try a geometric approach.

$\dot{x} = \sin x$ lets plot $\langle \dot{x}, x \rangle$

this 1D so, it resembles flow on a line.



$\dot{x} > 0$ flow to right

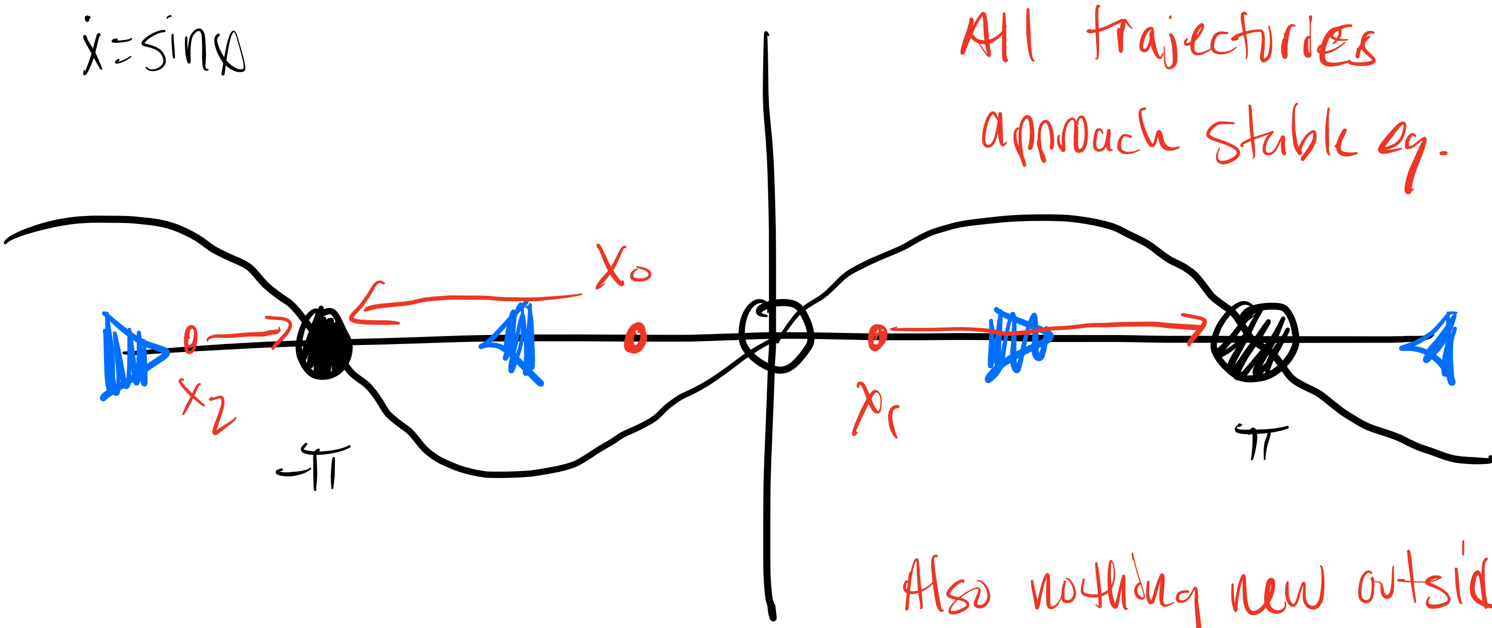
$\dot{x} < 0$ flow to left

$\dot{x} = 0$ defines critical pts.

Now we can see what happens depending on the initial condition. Every trajectory eventually ends its motion at a stable pt.

$\dot{x} = \sin x$

All trajectories approach stable eq.



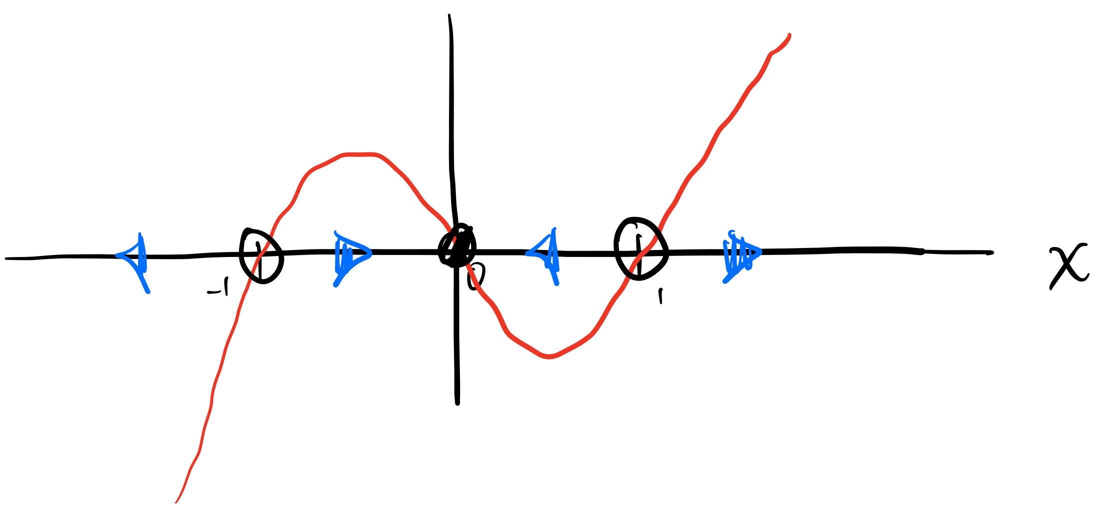
Also nothing new outside $[-\pi, \pi]$

Awesome, let's try another example,

Example:

$$\dot{x} = x^3 - x = (x^2-1)x$$

$\dot{x} = 0$ critical pts $x = \pm 1, 0$



for $|x_0| > 1$ solutions run off to $|x| \to |\infty|$

otherwise all end @ $x = 0$ eventually

$$\dot{x} = x^3 - x \Rightarrow \frac{dx}{x^3-x} = dt$$

$$\int_0^t dt' = \int_{x_0}^{x(t)} \frac{dx'}{x'^3 - x}$$

$$t = \frac{1}{2}\ln(1-x'^2) - \ln(x') \Big|_{x_0}^{x(t)}$$

$$t = \left(\frac{1}{2}\ln(1-x^2) - \ln(x)\right) - \left(\frac{1}{2}\ln(1-x_0^2) - \ln(x_0)\right)$$

again oof.

We still need some tools to deal with find trajectories!

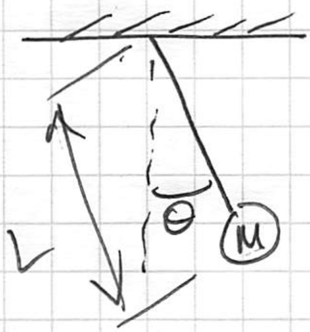
Let's pause for a minute and appreciate what we have done.

without solving the differential eq, we could characterize all the possible solutions!

The analogy of flow on a line can be extended to 2D when we have 2nd order ODEs.

The Harmonic Oscillator gets a bad rap. (1)

Consider a Pendulum with mass, M, & length, L



Through a variety of analyses we can show that

$$ML\ddot{\theta} = -gM\sin\theta$$

or

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

we often very quickly limit ourselves to small oscillations (i.e., θ small) so that $\sin(\theta) \simeq \theta$ and thus,

$$\ddot{\theta} \simeq -g/_L\,\theta \qquad \text{with } \omega^2 = g/_L$$

then

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

This approximation gets a bad rap b/c it's widely useful (in "limited" contexts):

→ circuits w/ inductor and cap: $\ddot{Q} = -\frac{1}{CL}Q$

→ spring mass: $\ddot{x} = -k/_m\,x$

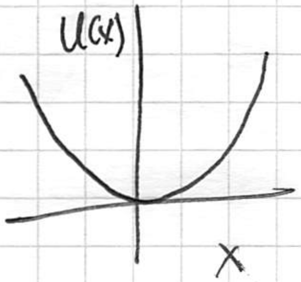
→ water in U tube: $\ddot{y} = -\frac{2g\rho A}{M}y$

→ jump rope: $\ddot{u} = -\frac{T}{\lambda}\left(\frac{n\pi}{d}\right)^2 u$

etc!

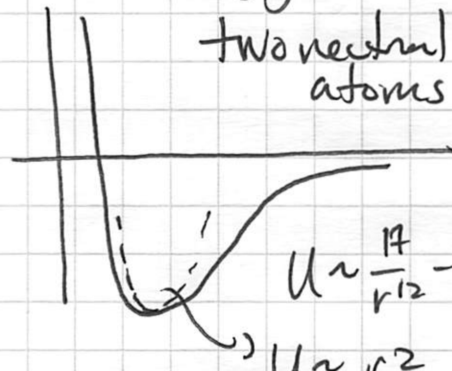
Anything with linear restoring force

② But also, nature tends to be energy minimizing.



$U = \frac{1}{2}kx^2$

Harmonic Oscillator $\Rightarrow$



$U \sim \frac{A}{r^{12}} - \frac{B}{r^6}$

$U \sim r^2$

Gives great help near energy minima. $\Leftarrow$ Boom! SHO near energy minimum

What if we want more though?

Let's go back to the pendulum,

$\ddot{x} = -\sin x$ where I've absorbed ω^2 in time (or set to 1)

What do we do?

find $x(t)$? But $x(t)$ depends very much on x_0 & v_0. So could get many trajectories ($x(t)$'s)

Enter Dynamical Systems!

- → don't solve for specifics
- → characterize lots of solutions @ once
- → look for qualitatively different behavior

Let's go back to the approximate form,

$\ddot{x} = -x$

We can begin to characterize a whole bunch of solutions by considering a phase space.

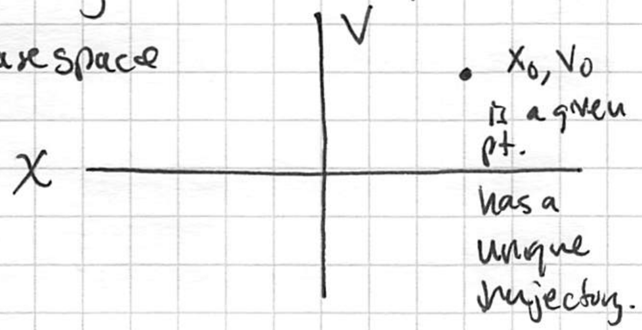
③ Phase Space

→ a space in which all possible states of a system can be shown.

⇒ each state is a unique pt in the space.

For a second order differential equation, we only need two points x & v (think: x_0, v_0 known means $x(t)$ is a unique soln)

phase space



Change differential equation b/c we need one for x & one for v.

One 2^{nd} order ODE ⟺ Two Coupled 1^{st} order ODEs.

$$\ddot{x} = -x \quad \Longleftrightarrow \quad \begin{aligned} \dot{v} &= -x \\ \dot{x} &= v \end{aligned}$$

Map this to phase space

$$\langle \dot{x}, \dot{v} \rangle = \langle v, -x \rangle$$

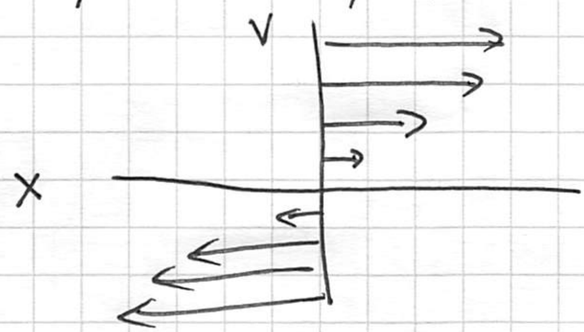
how x & v change ~ where you are in space

Consider: $x = 0$ line.

$$\langle \dot{x}, \dot{v} \rangle = \langle v, 0 \rangle$$

→ v does not change

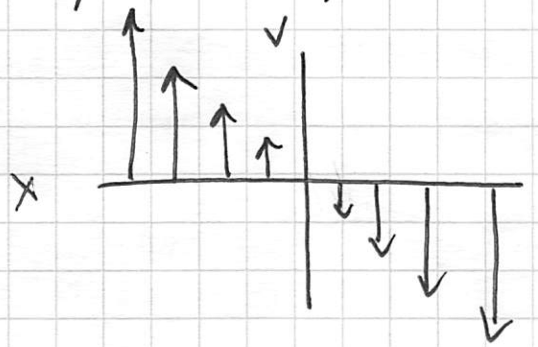
→ x changes as v changes (same sign)



(4) Consider: $v=0$ line

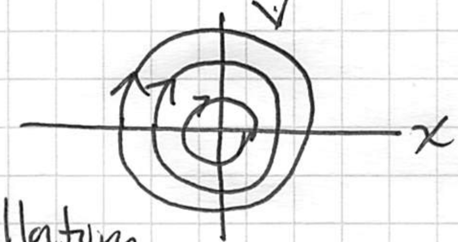
$\langle \dot{x}, \dot{v} \rangle = \langle 0, -x \rangle$ → x does not change

→ v changes as x does (negative sign)



$\Longrightarrow$

Put it together and connect "the dots"...



Cool! ✓ All solutions are oscillatory

✓ Total energy is conserved

✓ Each loop characterizes all initial conditions with given energy

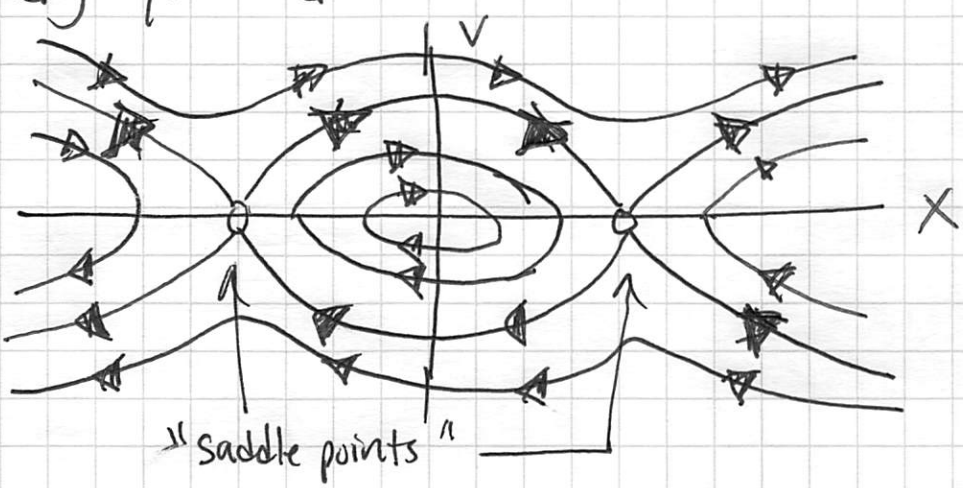
But! At some point this breaks down, energy is so large small oscillations is no good!

and then we have $\ddot{x} = -\sin x$ 2nd order

Or, $\dot{v} = -\sin x$ 2 1st order

$\dot{x} = v$ coupled ODEs

At low energy structure looks similar, but high energy quite different.



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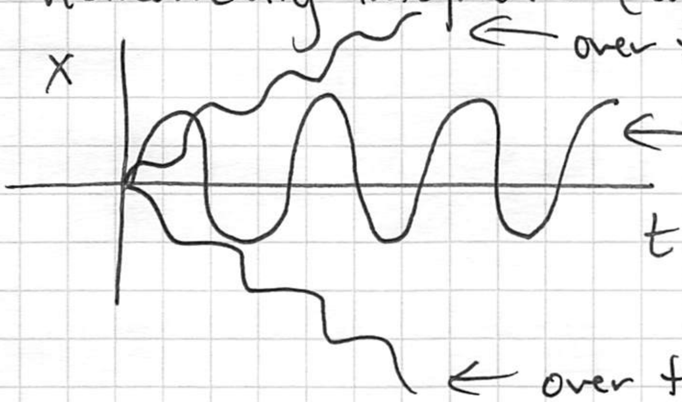
We get new families of solutions!

1. periodic, but not sinusoidal
   $\Rightarrow$ for very small x & v $\rightarrow$ near sinusoidal
2. clockwise rotations over the top (v always >0)
3. counterclockwise rotations over the top (v always <0)

Ok but what about specific trajectories?

numerically integrate (e.g. ODEINT)



What about Damped Motion?

$\ddot{x} = -b\dot{x} - \sin x$ approx $\ddot{x} = -b\dot{x} - x$

same approach,

Exact Phase Space

$\dot{v} = -bv - \sin x$

$\dot{x} = v$

Approx

$\dot{v} = -bv - x$

$\dot{x} = v$

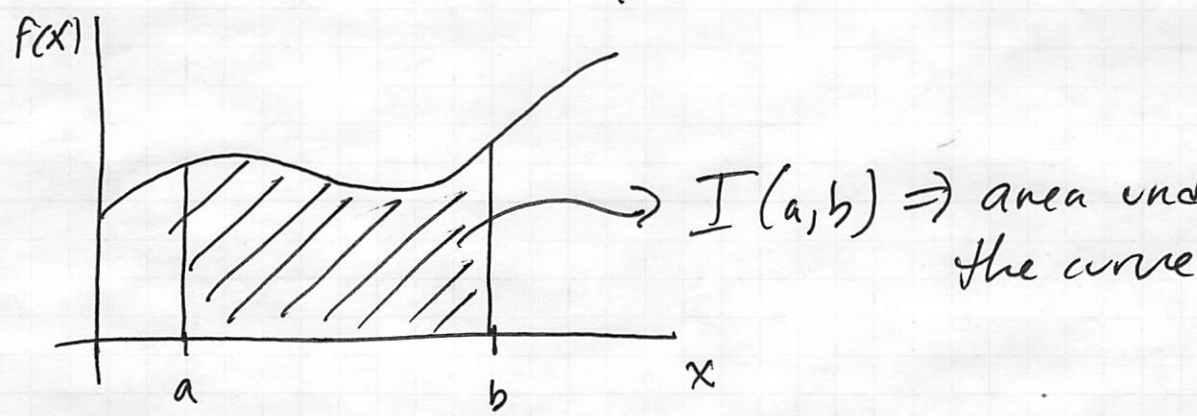
new phenomenon an attractor!

Up to now, your primary experience has been with integrals of functions with known anti-derivatives → that is, analytical integrals.

However, many functions don't have analytical anti-derivatives, but the concept of an integral is still there (i.e., the area under a curve).

Suppose we have some function $f(x)$ for which we want to compute it's integral between a & b,

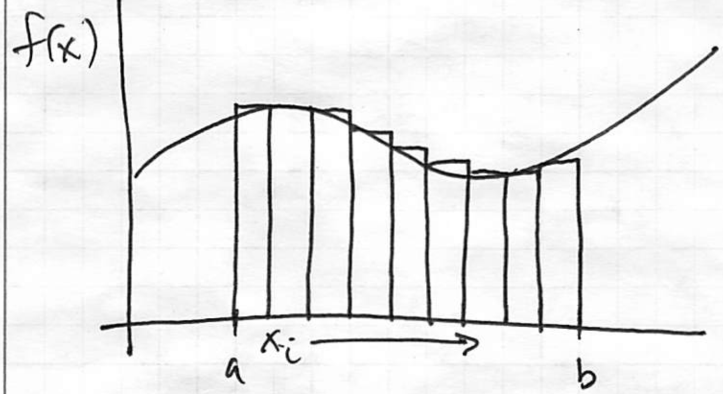
$$I(a,b) = \int_a^b f(x)\,dx$$



If we are unable to compute this integral because there's no analytic anti-derivative of the function*, $f(x)$, we can do it numerically by estimating the area under the curve.

* this technique also works for $f(x)$ where $\int f(x)\,dx$ is known.

Perhaps the simplest approach, which you've already thought of is using small ~~rectabl~~ rectangles



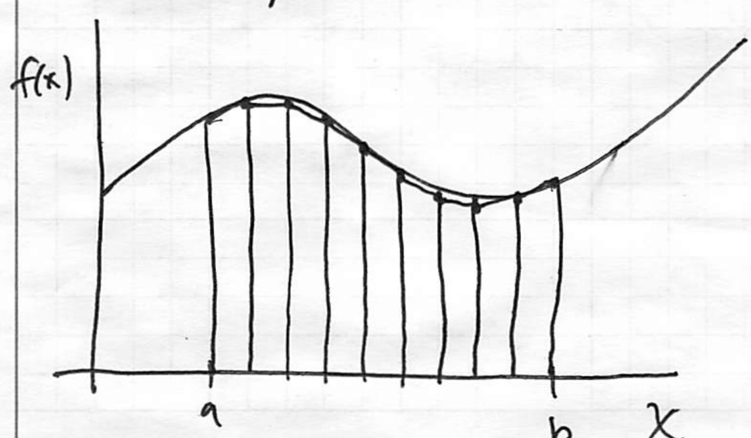
Use equally spaced rectangles with one edge (first or last) equal to $f(x_i)$ at each x_i.

We can then add up the total area using the sum of the areas of each rectangle.

⇒ This gives a poor approximation of the integral as it only takes into account the value of the function. We can do slightly better (often quite a bit better) by taking into account the value and the slope of $f(x)$.

Trapezoidal Rule

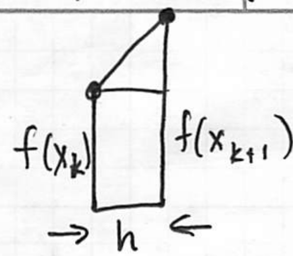
If we instead take into account the approximate slope between neighboring points, we get (a much) better approximation (use Trapezoids)



Use equally spaced trapezoids instead of rectangles and add them up as before.

Area of a Trapezoid:

Area of rectangle +
Area of triangle.



$$= f(x_k)h + \frac{1}{2}\left[f(x_{k+1}) - f(x_k)\right]h$$

$$A_{k+1} = \frac{1}{2}\left[f(x_k) + f(x_{k+1})\right]h$$

this makes things pretty straight-forward. Each slice contributes an amount equal to A_{k+1}.

So let h be the width of the slices where, $h = (b-a)/N$ where N is the # of slices.

For the kth slice, the right hand side is at $x_k = a + kh$ and the left hand side is at $x_{k-1} = a + kh - h = a + (k-1)h$

Trapezoidal Rule: **Area of k^{th} slice**

$$A_k = \frac{1}{2}h\left[f(a+(k-1)h) + f(a+kh)\right]$$

Approximating our integral

So we just add up all the contributions,

$$I(a,b) \simeq \sum_{k=1}^{N} A_k = \sum_{k=1}^{N} \frac{1}{2}h\left[f(a+(k-1)h + f(a+kh)\right]$$

$$= h\left[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(N-1)h) + \frac{1}{2}f(b)\right]$$

$$= h\left[\frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{N-1} f(a+kh)\right] \leftarrow \text{algorithm ready!}$$

Example: known analytical integral

$f(x) = x^4 - 2x + 1$ from $x=0$, to $x=2$,

$$I(0,2) = \int_0^2 x^4 - 2x + 1 = \frac{1}{5}x^5 - x^2 + x \Big|_0^2 = 4.4$$

Let's use 10 slices, (Live Code this)

```
def f(x):
    return x**4 - 2*x + 1

N=10
a=0
b=2.0
h=(b-a)/N
s=0.5*f(a) + 0.5*f(b)    # ½ f(a) + ½ f(b)
for k in range(1,N):
    s+=f(a+k*h)    # Σ_{k=1}^{N} f(a+kh)
print(h*s)
```

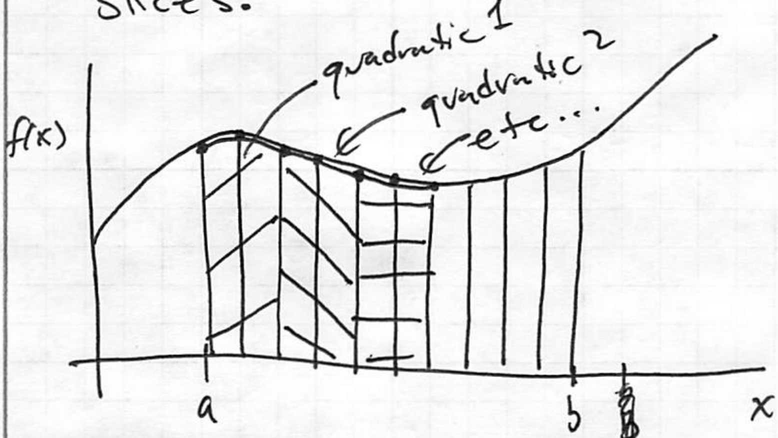
Result = 4.50656   increase steps $\Rightarrow$ 4.40001 ($N=1000$)

the trapezoidal rule work relatively well for many cases, but it can be slow (i.e. need alot of steps) and is inherently less acurrate than more advanced approaches.

It only takes into account value and slope. It's a "first-order" integration method. It is only accurate up to terms proportional to h (step size). Errors are h^2 and higher.

Simpson's Rule

A better method will use value, slope, and approximate curvature. We can use quadratic functions to approximate the area of two adjacent slices.



Suppose we have three points $x = -h, 0, +h$ and we try to fit a quadratic to those points

$$Ax^2 + Bx + C$$ so,

$$f(-h) = Ah^2 - Bh + C \qquad f(0) = C \qquad f(+h) = Ah^2 + Bh + C$$

We can solve these for the unknown coeffs,

$C = f(0)$ (easy.)

$$A = \frac{1}{h^2}\left[\frac{1}{2}f(-h) - f(0) + \frac{1}{2}f(h)\right]$$

$$B = \frac{1}{2h}\left[f(h) - f(-h)\right]$$

The area under that quadratic approximation is,

$$\int_{-h}^{h}(Ax^2 + Bx + C)\,dx = \frac{2}{3}Ah^3 + 2Ch = \frac{1}{3}h\left[f(-h) + 4f(0) + f(h)\right]$$

This result is Simpson's rule and is very powerful b/c it only depends on the value of the function at 3 equally spaced points.

So for the pair of adjacent "bins," the area would be

$$\text{Area} \simeq \frac{1}{3}h\left[f(x_k) + 4f(x_{k+1}) + f(x_{k+2})\right]$$

The total integral is the sum of these pair of bins,

$$\begin{aligned} I(a,b) \simeq\ & \frac{1}{3}h\left[f(a) + 4f(a+h) + f(a+2h)\right] \\ & + \frac{1}{3}h\left[f(a+2h) + 4f(a+3h) + f(a+4h)\right] + \dots \\ & + \frac{1}{3}h\left[f(a+(N-2)h) + 4f(a+(N-1)h) + f(\overset{b}{\cancel{a+Nh}})\right] \end{aligned}$$

We can clean this up by collecting terms,

$$I(a,b) \simeq \frac{1}{3}h\Big[f(a) + \underbrace{4f(a+h)}_{\substack{\text{odd terms} \\ \times 4}} + \underbrace{2f(a+2h)}_{\substack{\text{even terms} \\ \times 2}} + 4f(a+3h) + \dots + f(b)\Big]$$

$$I(a,b) \simeq \frac{1}{3}h\left[f(a) + f(b) + 4\sum_{k\,\text{odd}}^{N-1} f(a+kh) + 2\sum_{k\,\text{even}}^{N-2} f(a+kh)\right]$$

"trick" in python

odd terms: for k in range(1, N, 2) ← take 2

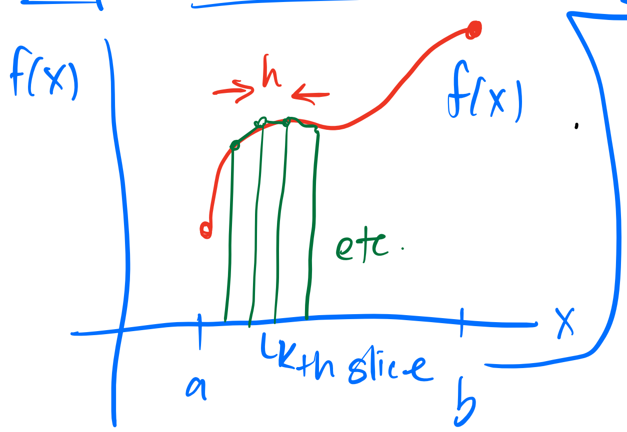
even terms: for k in range(2, N, 2) ← steps

Typically Simpson's rule is much better (more efficient and more accurate) than the Trapezoidal rule. It's a third order method → accurate to h^3 with error terms of h^4 and higher.

Integrating ODEs

Numerical Integration extends beyond the use to perform analytical integrals,

Trapezoidal Rule

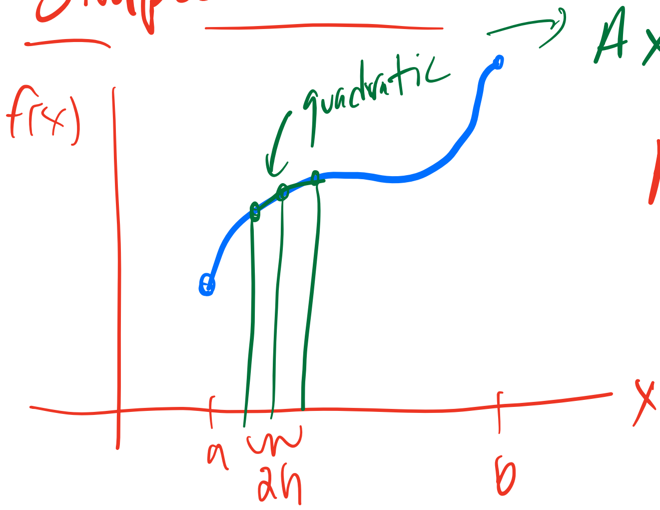


$$\text{Area}_k = \frac{1}{2}h\Big[f(a+(k-1)h) + f(a+kh)\Big]$$

$$I \simeq \sum_k A_k$$

$$I = h\left[\frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{N-1} f(a+kh)\right]$$

Simpson's Rule



$$Ax^2 + Bx + C$$

$$A_k = \frac{1}{3}h\Big(f(x_k) + 4f(x_{k+1}) + f(x_{k+2})\Big]$$

$$I \simeq \sum_k A_k$$

$$I \simeq \frac{1}{3}h\left[f(a) + f(b) + 4\sum_{k\ \text{odd}}^{N-1} f(a+kh) + 2\sum_{k\ \text{even}}^{N-1} f(a+kh)\right]$$

We can use these to "integrate the equations of motion."

First Order Differential Equation

Let's see how this works for a 1st order ODE,

$$\frac{dx}{dt} = f(x,t) \quad \text{or} \quad \dot{x} = f(x,t)$$

Cool. Let's say we know where we are a time, t, and we want to predict (estimate) where we will be a short time, h, later.

The standard approach involves a Taylor expansion around t,

$$x(t+h) = x(t) + h\frac{dx}{dt} + \frac{1}{2}h^2\frac{d^2x}{dt^2} + \ldots$$

$$= x(t) + h\frac{dx}{dt} + O(h^2)$$

order h^2 and above terms.

So our linear (in h) approx gives,

$$x(t+h) = x(t) + h\frac{dx}{dt} \quad \text{or,}$$

$$\boxed{x(t+h) = x(t) + h f(x,t)}$$

Euler integral.

Great! So for first order ODEs we can use this!

2nd Order ODEs?

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) \quad \text{or} \quad \ddot{x} = f(x, \dot{x}, t)$$

we make two first order ODEs,

let $v = \frac{dx}{dt}$ then $\frac{dv}{dt} = \frac{d^2x}{dt}$

So that,

$$\frac{dv}{dt} = f(x, v, t) \quad \text{and} \quad \frac{dx}{dt} = v$$

then like before,

$$v(t+h) = v(t) + h f(x, v, t)$$
$$x(t+h) = x(t) + h v(t)$$

Euler for 2nd order

!! This approach does not conserve energy ↑

It will have big issues long term or for oscillations.

It is corrected by Cromer (1963)

$$v(t+h) = v(t) + h f(x, v, t)$$

$$x(t+h) = x(t) + h \underbrace{v(t+h)}$$

take the prediction from prior step & use here

Euler-Cromer Integration 2nd Order

$$v(t+\Delta t) = v(t) + \frac{F(x,v,t)}{m} \Delta t$$

← net force

$\vec{F}_{net} = m\vec{a}$

$$x(t+\Delta t) = x(t) + v(t+\Delta t)\Delta t$$