CW5 – Potential Energy

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The work energy theorem related the change in Kinetic Energy to the Work done on / by the system,

$$\Delta K_{sys} = W .$$

We found that the work done by any force is given by,

$$W = \int_C \vec{F} \cdot d\vec{r}$$

where C is the particular path taken

For a particular kind of force that integral is path independent.

Conservative Forces

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There are a number of ways to define a conservative force, each can be derived from the other.

(i) $\nabla \times \vec{F} = 0 \Leftarrow$ only $\vec{F}(\vec{r})$ no $\vec{F}(\dot{\vec{r}})$

(ii) $\oint_C \vec{F} \cdot d\vec{r} = 0 \Leftarrow$ no energy change around any closed loop.

(iii) $\int_C \vec{F} \cdot d\vec{r}$ is path independent $\Leftarrow$ the intergral depends only on the end pts.

These properties suggest we define a quantity dependent on location whose difference is proportional to the work integral

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Define Potential Energy

$$\Delta PE = U(\vec{r}_f) - U(\vec{r}_i) = \Delta U = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}(\vec{r}) \cdot d\vec{r}$$

In 1D,

$$\Delta U_{sys} = -\int_{x_A}^{x_B} F(x)\,dx$$

The prior properties and the above definition give the following relationship

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r})$$

where

$$\nabla = \hat{x}\frac{d}{dx} + \hat{y}\frac{d}{dy} + \hat{z}\frac{d}{dz}$$

- One core definition of a conservative force guarantees that an appropriately defined potential, $U(x)$, leads to a conservative force. (4)

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r})$$

$$\nabla \times \vec{F} = \nabla \times (-\nabla U) = -\nabla \times \nabla U = 0$$

$$\boxed{\nabla \times \nabla f(\vec{r}) = 0}$$

$\Leftarrow$ the curl of a gradient is zero

Proof:

$$\nabla f(\vec{r}) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

Let's focus on $(\nabla \times \nabla f)_x$,

$$(\nabla \times \nabla f)_x = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}$$

partial derivative operator is linear so,

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

$$= 0$$

true for every component

$$\nabla \times \nabla f(\vec{r}) = 0$$

Returning to the workenergy theorem

$$\Delta K_{sys} = W = \int_C \vec{F} \cdot d\vec{r}$$

$\leftarrow$ conservative force?

$$\Delta K_{sys} - \int_C \vec{F} \cdot d\vec{r} = 0$$

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$$\boxed{\Delta K_{sys} + \Delta U_{sys} = 0}$$

kinetic & potential energy eqn.

Common Forms of Writing this ↑

$$\Delta KE + \Delta PE = 0$$

$$\Delta K + \Delta U = 0$$

$$\Delta T + \Delta U = 0$$

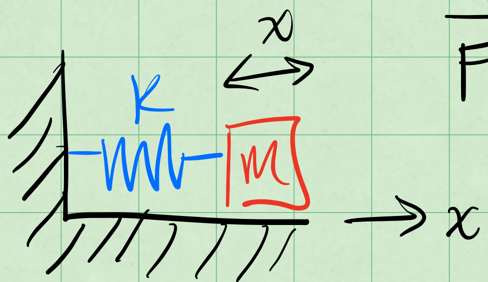
$$\Delta T + \Delta V = 0$$

most common and the form we will use in class.

Now what?

Let's explore some examples and investigate the potential landscape, the type of motion we expect, stability/instability, and classical turning points.

Example: Simple Harmonic Oscillator



$$\vec{F}(\vec{r}) = F(x)\,\hat{x}$$

$$F(x) = -kx$$

$$U(x) = -\int_0^x F(x')\,dx' = \frac{1}{2}kx^2$$

The SHO is a conservative system so that,

$$\Delta K + \Delta U = 0$$

or,

$$K + U = E_{tot} = \underline{\text{constant}}$$

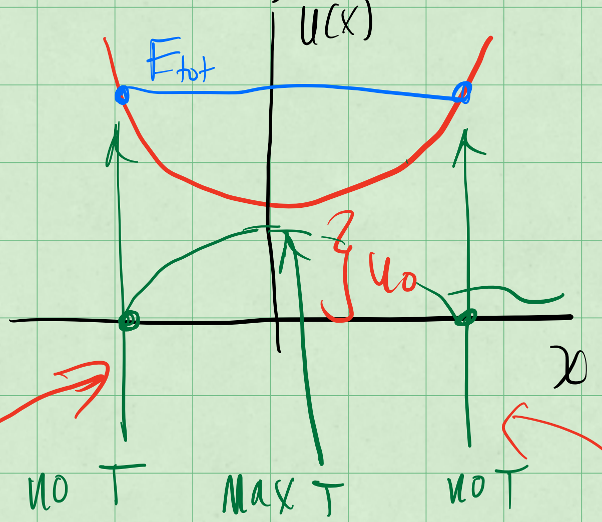
This means we can analyze the motion of the system by assuming a total amount of available energy.

$$K(v) + U(x) = E_{tot} \Rightarrow \underline{v(x)} \text{ or } \underline{x(v)}$$

@ E_{tot}

We can find phase trajectories of the motion at a given total energy, E_{tot}. 8

Let's graph $U(x)$,



$U(x) = \frac{1}{2}kx^2 + U_0$

only differences matter

No motion beyond here; classical turning points this shows that the SHO turns around at these maximum values for x. We also note the max T (max speed) is at $x=0$, where min U occurs.

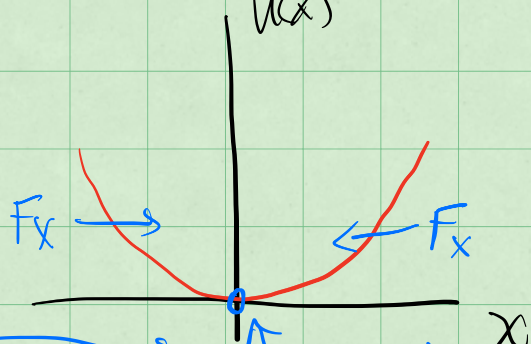
This motion is stable about $x=0$.

why?

Stability

Lets use the SHO example

$$U(x) = \frac{1}{2}kx^2$$



$$F(x) = -\frac{\partial U}{dx}$$

$$F(x) = -kx$$

forces push mass towards $x = 0$

special

$$\frac{dU}{dx} = 0 \Leftarrow$$ location?

Finding critical points

$x = 0$ for $U = \frac{1}{2}kx^2$ is a "critical point" a point where $F_x = 0$. We can find those points, x^*, by taking the first derivative of the potential and solving the resulting equation.

$U = \frac{1}{2}kx^2$ $\quad$ $\frac{dU(x^*)}{dx} = 0$ $\quad$ solve for x^*

$\frac{dU(x^*)}{dx} = kx^* = 0$ $\quad$ $x^* = 0$ as predicted before

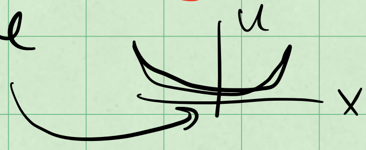
How do we see the fact that this point is stable?

$\Rightarrow$ we use the second derivative (curvature of $U(x)$ at x^*

$$\boxed{\frac{d^2U(x^*)}{dx^2} = k > 0}$$

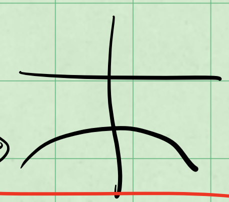
this is the condition for stability

upward facing curvature



so,

$$\frac{\partial^2 U(x^*)}{\partial x^2} < 0 \quad \text{unstable}$$



finally, $\partial^2 U(x^*)/\partial x^2 = 0 \Rightarrow$ flat (meta stable)

Summary for 1D potentials

- With $U(x)$, $F(x) = -dU/dx$
- Find critical points, x^*, using the first derivative.

$$\frac{dU(x^*)}{dx} = 0$$ solve for x^*

critical points

- With x^*, we can find stability,

$$\frac{d^2U(x^*)}{dx^2} > 0$$

stable

$$\frac{d^2U(x^*)}{dx^2} < 0$$

unstable

$$\frac{d^2U(x^*)}{dx^2} = 0$$

meta stable

Example: Morse Potential

The Morse potential is a model of diatomic atoms. ~~It's~~ a toy model and we will explore it in one dimension,

$$U(x) = D_e\left(1 - e^{-\alpha(x - x_e)}\right)^2$$

Here we have a modified spring with constant, k_e. The remaining coefficients are,

D_e : well depth

x_e : equilibrium bond length

$\alpha = \sqrt{k_e / 2D_e}$: parameter representing the spring & well depth

With

$$U(x) = D_e\left(1 - e^{-\alpha(x-x_e)}\right)^2$$

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$$F(x) = -\frac{dU}{dx}$$

$$= \frac{-d}{dx}\left(D_e\left(1-e^{-\alpha(x-x_e)}\right)^2\right)$$

$$= -2D_e\left(1-e^{-\alpha(x-x_e)}\right)\left(\alpha e^{-\alpha(x-x_e)}\right)$$

$$\boxed{F(x) = -2D_e\alpha\left(1-e^{-\alpha(x-x_e)}\right)e^{-\alpha(x-x_e)}}$$

Conservative force

$\nabla \times \vec{F} = 0$?

$\vec{F}(\vec{r}) = F(x)\hat{x} \rightarrow \nabla \times \vec{F}$?

$\hat{x}$ component get $\frac{\partial F_x}{\partial y}$ & $\frac{\partial F_x}{\partial z}$

Both are zero $\xrightarrow{b/c}$ $F_x(x)$

Stability? What is x^*? 14

$$\frac{dU(x^*)}{dx} = 0$$

$$\frac{dU(x^*)}{dx} = 2D_e\alpha\left(1-e^{-\alpha(x^*-x_e)}\right)e^{-\alpha(x^*-x_e)}$$

$$= 0$$

Both terms could vanish,

$$e^{-\alpha(x^*-x_e)} = 0 \quad (1)$$

$$\left(1-e^{-\alpha(x^*-x_e)}\right) = 0 \quad (2)$$

For eqn 1,

$$\lim_{x^*\to\infty} e^{-\alpha(x^*-x_e)} = 0 \qquad \boxed{x^*\to\infty}$$

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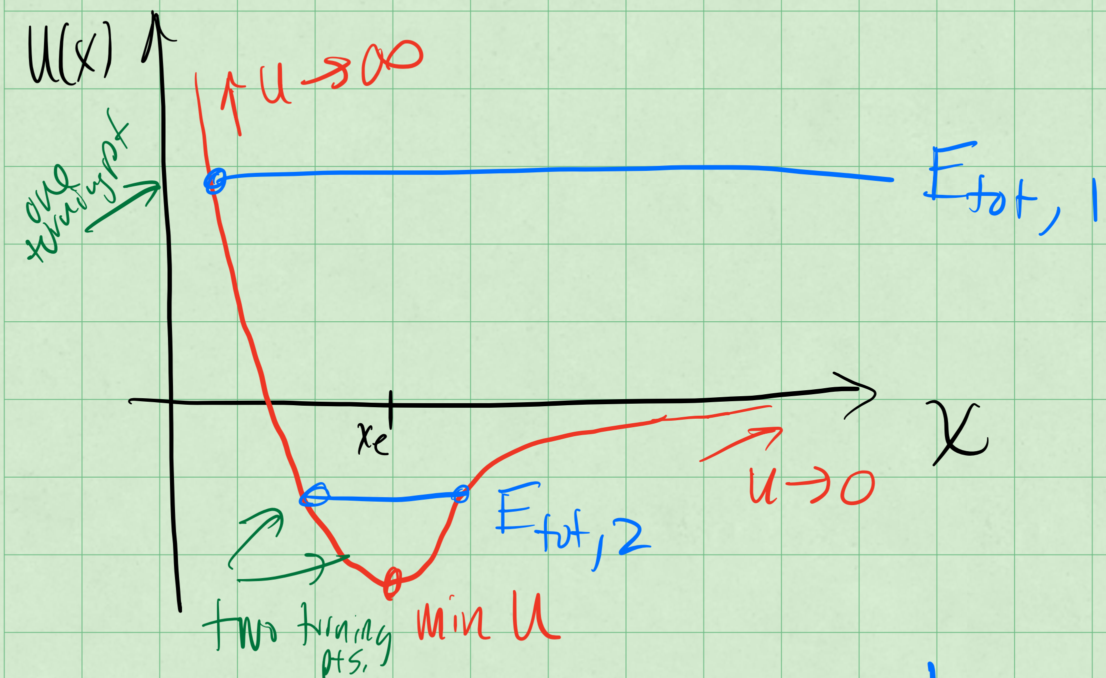
For eqn 2,

$$\left(1 - e^{-\alpha(x^* - x_e)}\right) = 0$$

$$-\alpha(x^* - x_e) = \ln(1) = 0$$

$$\boxed{x^* = x_e}$$

Two critical points. Why?



2 different kinds of motion!

$x^* = x_e$ looks stable, is it?

$$\frac{d^2U(x_e)}{dx^2} = ?$$

$$\frac{dU(x)}{dx} = 2D_e\alpha\left(1 - e^{-\alpha(x - x_e)}\right)e^{-\alpha(x - x_e)}$$

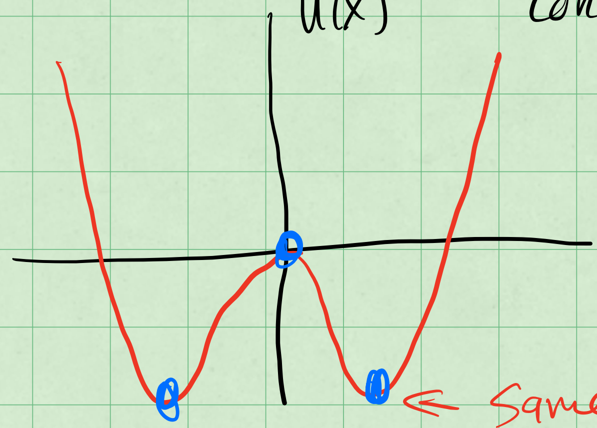
$$\frac{d^2U(x_e)}{dx^2} = 2D_e\alpha\left\{\left(\alpha e^{-\alpha(\overset{x_e}{x} - x_e)}\right)e^{-\alpha(\overset{x_e}{x} - x_e)} + \left(1 - e^{-\alpha(\overset{x_e}{x} - x_e)}\right)\left(-\alpha e^{-\alpha(\overset{x_e}{x} - x_e)}\right)\right\}$$

$$= 2D_e\alpha\left(\alpha + (1-1)(-\alpha)\right) = \underbrace{2D_e\alpha^2 > 0}_{\text{stable critical point, } x^* = x_e}$$

Example: Double Well Potential

$U(x) = ax^4 - bx^2$ where a & b are positive constants

$a, b > 0$



3 critical points!

same level

$$U(x) = ax^4 - bx^2$$

$$F(x) = -\frac{dU}{dx} = -4ax^3 + 2bx$$

What about these critical points, x^*?

$$dU(x^*)/dx = 0$$ & solve

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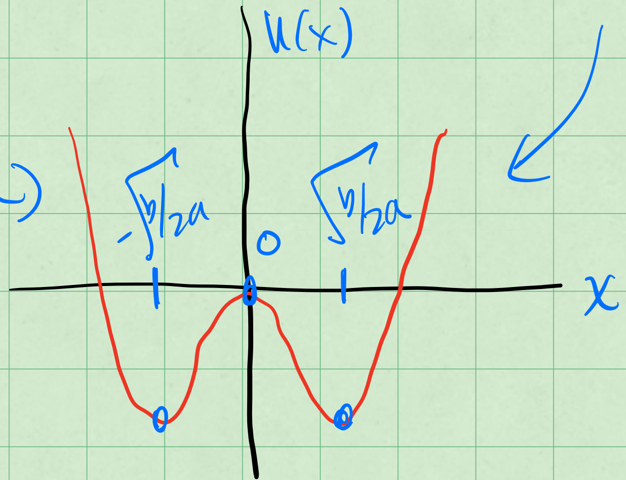
$$\frac{dU(x^*)}{dx} = 4ax^{*3} - 2bx^* = 0$$

$$x^*(4ax^{*2} - 2b) = 0$$

$$x^* = 0 \qquad x^* = \pm\sqrt{\frac{b}{2a}}$$

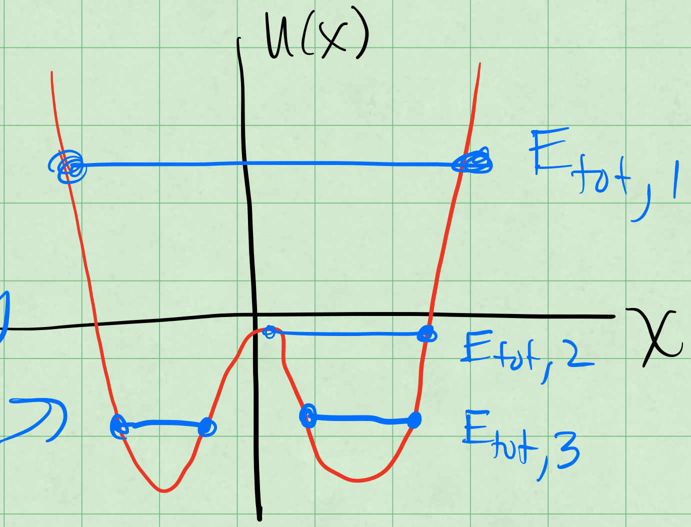


Stability? Let's just look at $x^* = 0$

$$\frac{d^2U(x^*)}{dx^2} = 12ax^{*2} - 2b$$

$$= -2b < 0 \qquad \text{unstable!}$$

Different motion and turning pts based of E_{tot}



Trajectories? $v(x)$; $x(v)$?

Pick an E_{tot},

$$T + U = E_{tot} = \text{constant}$$

$$\frac{1}{2} m v_x^2 + (ax^4 - bx^2) = E_{tot}$$

$$v_x(x) = \pm \sqrt{\frac{2(E_{tot} - ax^4 + bx^2)}{m}}$$

this puts a limit of the total

energy for this model to make sense,

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$V_x(x)$ must be real so,

$$E_{tot} - ax^4 + bx^2 \geq 0$$

$$E_{tot} \geq -bx^2 + ax^4$$

The location where $U(x)$ is smallest is

$$x^* = \pm\sqrt{b/2a}$$

$$E_{tot} \geq -b\left(\frac{b}{2a}\right) + a\left(\frac{b}{2a}\right)^2$$

$$\geq \frac{-b^2}{2a} + \frac{b^2}{4a} = \frac{-b^2}{4a}$$

$$\boxed{E_{tot} \geq \frac{-b^2}{4a} \quad \text{for given } a \,\&\, b}$$