Introduction to Signal Analysis

Now that we have seen we can use superposition to build solutions to the wave equation, we might ask can we use the concept of superposition to deal with more general (periodic) signals.

We've claimed that any $f(t)$ that is periodic can be written as,

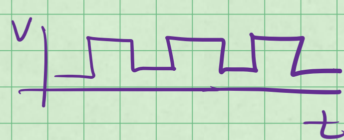
$$\chi(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right)$$

where $n\omega_0$ represent harmonics of some base periodicity, $\omega_0 \Leftarrow$ typically the lowest observed frequency $\rightarrow$ longest periodic signal.

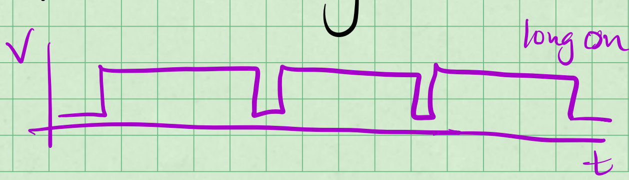
this is maybe best conducted via example

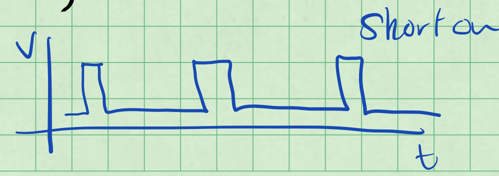
The Duty Cycle

A common signal in electronic systems is the duty cycle. A signal is turned off and on at some regular interval,

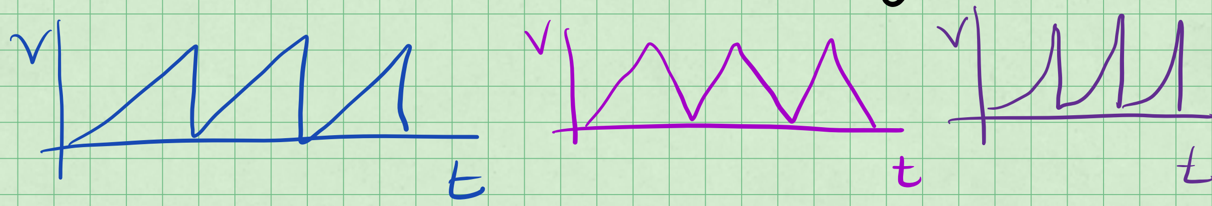


which can vary interval widths,





and can ramp up and down differently,



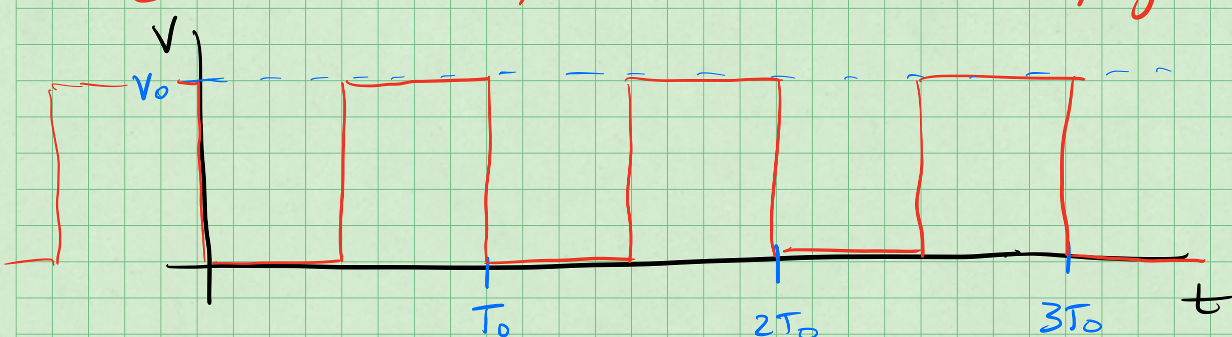
In all cases, we can use Fourier Decomposition

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

☆ later, we will use the fast fourier transform FFT to show how to get approximate a_n's & b_n's.

Example: The Square Wave

Consider the square wave (50% duty cycle),



Mathematically this signal is discontinuous,

$$V(t) = \begin{cases} 0 & 0 < t \leq T_0/2 \\ V_0 & \frac{T_0}{2} < t < T_0 \end{cases}$$

over one cycle

Orthogonal Functions

We've encountered orthogonality before when we took scalar (dot) products.

* vectors $\vec{a}$ & $\vec{b}$ are orthogonal if

$$\vec{a} \cdot \vec{b} = 0$$

Functions are orthogonal over an interval (a to b),

$$\int_a^b A(x)\,B(x)\,dx = 0$$

is the defn of orthogonal for two real valued functions

Aside: if A & B are complex then $\int_a^b A^*(x)\,B(x)\,dx = 0$

It is possible to have sets of orthogonal functions, let $A_n(x)$ be defined as an orthogonal set for $n = 1, 2, 3, \ldots$

$$\int_a^b A_n^*(x)\, A_m(x) = \begin{cases} 0 & n \neq m \\ \text{constant} \neq 0 & n = m \end{cases}$$

We have already seen this with sinusoidal functions,

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx) = \begin{cases} 0 & n \neq m \\ \pi & n = m \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx) = \begin{cases} 0 & n \neq m \\ \pi & n = m \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx) = 0 \quad \text{for any } n \text{ or } m$$

Complex: $$\int_{-\pi}^{\pi} (e^{inx})^* e^{imx}\, dx = \int_{-\pi}^{\pi} e^{-inx} e^{imx}\, dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

Back to Fourier Analysis

$$V(t) = \begin{cases} 0 & 0 < t < T_0/2 \\ V_0 & T_0/2 < t < T_0 \end{cases} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$V(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

Example to extract a_n's

$$\int_0^{T_0} V(t) \cos(m\omega_0 t)\, dt = \int_0^{T_0} \left(\frac{a_0}{2} + \sum \left(\quad \right)\right) \cos(m\omega_0 t)\, dt$$

orthogonality gives only $\neq 0$ when $n = m$,

$$\int_0^{T_0} V(t) \cos(n\omega_0 t) = \int_0^{T_0} a_n \cos^2(n\omega_0 t)\, dt$$

$$a_n \underbrace{\int_0^{T_0} \cos^2(n\omega_0 t)\,dt}_{\text{can solve}} = \underbrace{\int_0^{T_0} v(t)\cos(n\omega_0 t)}_{\text{need } v(t)}$$

$$\int_0^{T_0} \cos^2\left(\frac{n2\pi t}{T_0}\right) dt$$

$$= \frac{1}{2}\left[\int_0^{T_0} \left(1 + \cos\left(\frac{4\pi n t}{T_0}\right)\right) dt\right]$$

$$= \frac{1}{2}\left(t + \frac{T_0}{4\pi n}\sin\left(\frac{4\pi n t}{T_0}\right)\right)\Bigg|_0^{T_0}$$

$$= \frac{1}{2}\left(T_0 + \cancelto{0}{\frac{T_0}{4\pi n}\sin(4\pi n)} - \cancelto{0}{0} - \cancelto{0}{\frac{T_0}{4\pi n}\sin(0)}\right)$$

$$\int_0^{T_0} \cos^2\left(\frac{2n\pi t}{T_0}\right) = \frac{T_0}{2}$$

Thus,

$$a_n \frac{T_0}{2} = \int_0^{T_0} v(t)\cos(n\omega_0 t)\,dt$$

or,

$$a_n = \frac{2}{T_0}\int_0^{T_0} v(t)\cos(n\omega_0 t)\,dt$$

similarly,

$$b_n = \frac{2}{T_0}\int_0^{T_0} v(t)\sin(n\omega_0 t)\,dt$$

We will use Fourier's Trick to find a_n's & b_n's to approximate this duty cycle with continuous functions,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$\boxed{\begin{aligned} a_n &= \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt \qquad n \neq 0 \\ b_n &= \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt \end{aligned}}$$

using orthogonal functions to get coeffs.

Let's look at it first and compare to the model.

1. $V(t)$ is not symmetric so a_n's $= 0$. (follows sine symmetry)

2. $\frac{a_0}{2}$ is just the DC offset (average of signal)
   $\hookrightarrow a_0 = V_0$

<u>Our simplified model</u>

$$f(t) = \frac{V_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

All that's left is to find b_n,

$$b_n = \frac{2}{T_0}\int_0^{T_0} v(t)\sin(n\omega_0 t)\,dt \quad ; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$v(t) = \begin{cases} 0 & 0 < t < T_0/2 \\ V_0 & T_0/2 < t < T_0 \end{cases}$$

Our integral simplifies to,

$$b_n = \frac{2}{T_0}\int_{T_0/2}^{T_0} V_0 \sin\left(n\frac{2\pi t}{T_0}\right)dt = \underbrace{\frac{2V_0}{T_0}\int_{\frac{T_0}{2}}^{T_0} \sin\left(\frac{2\pi n}{T_0}t\right)dt}$$

$$\left(\frac{T_0}{2\pi n}\right)\left(\cos(n\pi) - \underbrace{\cos(2n\pi)}_{1}\right)$$

$$b_n = \frac{2V_0}{T_0}\left(\frac{T_0}{2\pi n}\right)\left(\cos(n\pi) - 1\right)$$

$$\boxed{b_n = \frac{V_0}{n\pi}\left[\cos(n\pi) - 1\right]}$$

n	b_n
1	$-\frac{2V_0}{\pi}$
2	0
3	$-\frac{2V_0}{3\pi}$
4	0
5	$-\frac{2V_0}{5\pi}$
⋮	

$$v(t) = \frac{V_0}{2} + \sum_{n=1}^{\infty} \frac{V_0}{n\pi}\left[\cos(n\pi) - 1\right]\sin\left(\frac{2\pi n}{T_0}t\right)$$

!!!

But what does this look like?

Apply Fourier Series to the DDO

Let's go back to to Eom:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

Let $x(t)$ be the long term solution what we've called $x_p(t)$ in the past.

Define $D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$

a linear differential operator. So that,

$$Dx = f$$

Assume we have two different problems with different drivers, f_1 & f_2

Then,

$$Dx_1 = f_1 \qquad Dx_2 = f_2$$

$x_1(t)$ & $x_2(t)$ are the particular long term solutions for $f_1(t)$ & $f_2(t)$ driving.

If $f(t) = f_1(t) + f_2(t)$, then,

$$x(t) = x_1(t) + x_2(t) \text{ for } D.$$

$$Dx = D(x_1 + x_2) = Dx_1 + Dx_2 = f_1 + f_2 = f$$

For a collection of $f_n(t)$'s where

$f(t) = \sum_n f_n(t)$ we can define

$x(t) = \sum_n x_n(t)$ such that

$$D x_n(t) = f_n(t)$$

This becomes very useful if the force can be Fourier Decomposed, e.g.,

for an even function,

$$f(t) = \sum_n f_n \cos(n\omega t)$$

where $\omega = 2\pi/T$ T is the base period.

We can the construct long term solutions,

$$x_n = A_n \cos(n\omega t - \delta_n)$$

where A_n & δ_n are similar to our old solutions for $x_p(t)$

$$A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}}$$

$$\delta_n = \arctan\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$

Thus, long term solutions are of the form,

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n) \quad \text{with } \omega = \frac{2\pi}{T}$$

Finding long term solutions is just:

① Find f_n for given $f(t)$

② Compute A_n & δ_n

③ Write down $x(t) = \sum_n$