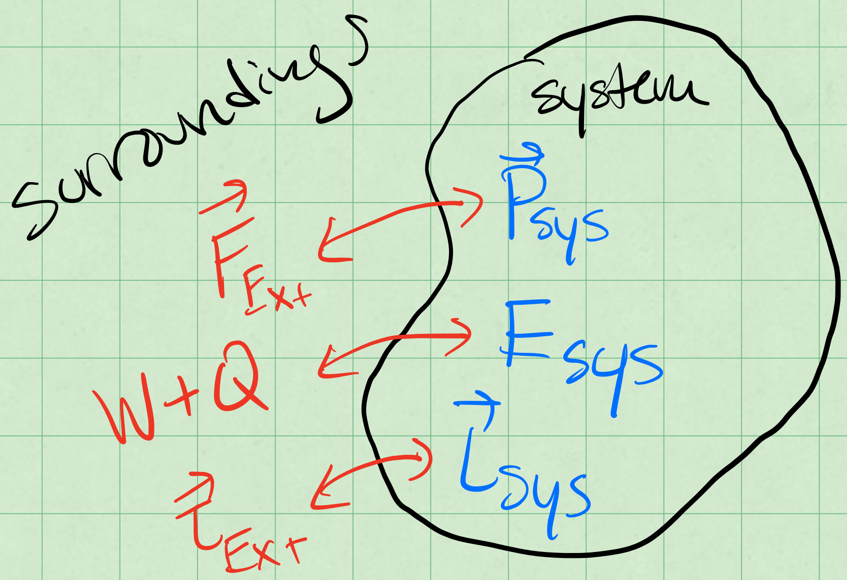
CW4 – Conservation Laws

(1)

You've seen conceptually that our view of mechanics is how a system interacts with its surroundings



These conceptual statements are codified by the following 3 governing principles:

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- Forces cause changes in momentum

$$\frac{d\vec{P}_{sys}}{dt} = \vec{F}_{Ext}$$

forces due to surroundings

$$\vec{P}_{sys,f} = \vec{P}_{sys,i} + \vec{F}_{Ext}\Delta t$$

- Work & Thermal Exchanges change the energy of a system

work done, heat exchanged

$$W + Q = \Delta E_{sys}$$

$$E_{sys,f} = E_{sys,i} + W + Q$$

- Torques cause changes in angular momentum.

$$d\vec{L}_{sys}/dt = \vec{\tau}_{Ext}$$

$$\vec{L}_{sys,f} = \vec{L}_{sys,i} + \vec{\tau}_{Ext}\,\Delta t$$

We will explore each in turn, and we have already begun our study of linear momentum through Newton's Second Law,

$$\frac{d\vec{p}}{dt} = \vec{F}_{net} \Rightarrow m\vec{a} = \vec{F}_{net}$$

But let's start our exploration with energy.

The Work-Energy Theorem

Our model for kinetic energy stems from a series of experiments dropping stones into clay. The depth of the hole made was proportional to the square of the impact speed. 4

Through many experiments we have quantified the kinetic energy of a point particle,

$$K = \frac{1}{2} m (\vec{v} \cdot \vec{v})$$

(we also use $T = \frac{1}{2} m (\vec{v} \cdot \vec{v})$)

↳ this is the classical kinetic energy. as we know now, its the energy of motion in the limit that $v/c \ll 1$.

aside:

$$E_{tot} = \gamma m c^2 \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\frac{v}{c} = 0 \Rightarrow E_{tot} \to mc^2 = E_{rest}$$

$$E_{tot} - E_{rest} = K = (\gamma - 1) mc^2$$

$$K = (\gamma - 1) mc^2$$

(5)

when $\frac{v}{c} \ll 1$, $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v^4}{c^4}\right)$

$$\gamma \simeq 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$$

$$K\left(\text{when } \frac{v}{c} \ll 1\right) \simeq \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right)mc^2$$

$$K \simeq \frac{1}{2}mv^2 \quad \checkmark$$

Developing the Work-Energy Thm

let $K = \frac{1}{2}m(\vec{v}\cdot\vec{v})$ how does K change?

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}m(\vec{v}\cdot\vec{v})\right)$$

assume point particle

$$\frac{dK}{dt} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{m}{2}\underbrace{\left(\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}\right)}_{2\frac{d\vec{v}}{dt}\cdot\vec{v}}$$

$$\frac{dK}{dt} = m\frac{d\vec{v}}{dt}\cdot\vec{v}$$

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$$\frac{dK}{dt} = \vec{F} \cdot \vec{v}$$

let's discretize K to make sense of this,

$$\frac{\Delta K}{\Delta t} = \vec{F} \cdot \vec{v}$$

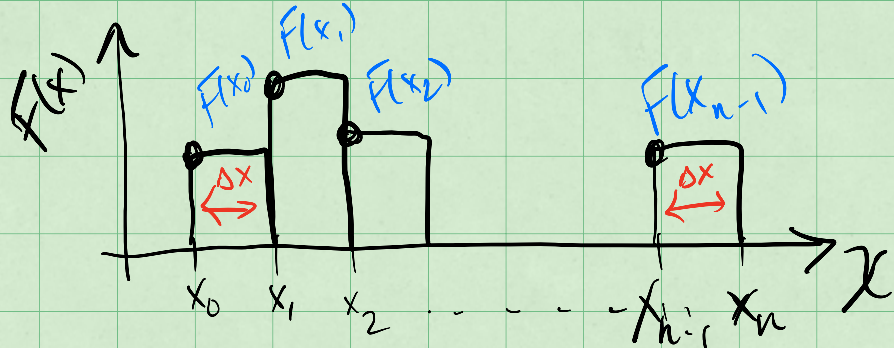
$$\Delta K = \vec{F} \cdot \vec{v} \Delta t$$

displacement of the body in a time Δt.

$$\Delta K = \underbrace{\vec{F} \cdot \Delta \vec{r}}$$

Work done in a displacement $\Delta \vec{r}$ by a force $\vec{F}$.

Discrete Model (1D)



$$x = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

$$\Delta K = \sum_{i=0}^{n-1} F(x_i)\Delta x = K_f - K_i$$

$$= \frac{1}{2}mv_n^2 - \frac{1}{2}mv_0^2$$

$$\lim_{\Delta x \to 0} \sum_{i=0}^{n-1} F(x_i)\Delta x = \int_{x_0}^{x_n} F(x)\,dx$$

Continuous Model

$$\Delta K = \text{Work Done} = \underbrace{\int_{x_0}^{x_f} F(x)\,dx}_{\text{one dimensional}}$$

What about more than 1D?

Consider a discretized Path, C

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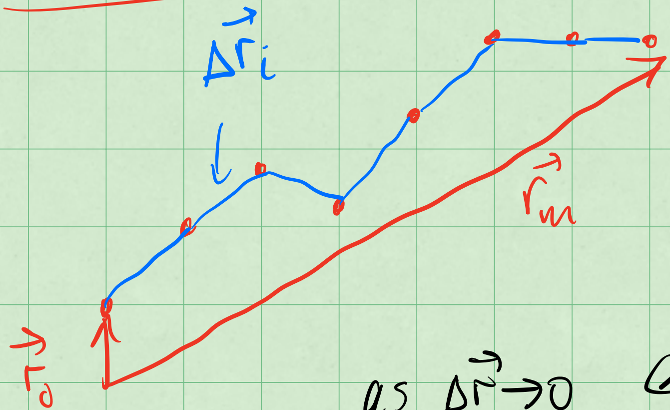


$$W = \sum_{\text{along } C} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

as $\Delta\vec{r} \to 0$

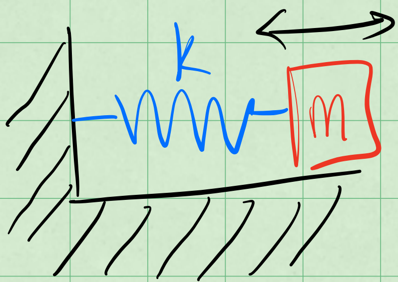
$$\boxed{W = \int_C \vec{F}(\vec{r})\cdot d\vec{r} \quad \text{path integral of } \vec{F} \text{ along } C}$$

Work can be $+$ $(\vec{F} \uparrow\uparrow \Delta\vec{r})$, $-$ $(\vec{F} \uparrow\downarrow \Delta\vec{r})$
or 0 $(\vec{F} \perp \Delta\vec{r})$

If $\vec{F}$ is always $\perp \Delta\vec{r}$, what kind of motion is possible?

$\underline{\Delta K = 0}$

Example: Spring-Mass Model



$F = -kx$

allow the spring to move from x_0 to x_1.
that corresponds to a speed of v_0 & v_1 respectively.

WE thm.

$W = \Delta K$

$$\Delta K = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$

$$W = \int_{x_0}^{x_1} (-kx)\,dx = -k\int_{x_0}^{x_1} x\,dx$$

$$= -k\left(\frac{1}{2}x_1^2 - \frac{1}{2}x_0^2\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}kx_1^2 + \frac{1}{2}kx_0^2$$ hmm....

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$$

$$\underbrace{\frac{1}{2}mv_1^2}_{\text{energy of motion}} + \underbrace{\frac{1}{2}kx_1^2}_{\text{another energy}} = \text{constant!}$$

$\frac{1}{2}kx^2 \Rightarrow$ potential energy of a spring mass system.

$$\boxed{U_{spring} = \frac{1}{2}kx^2}$$

We use U & V for potential energy.

Total Energy = Kinetic + Potential

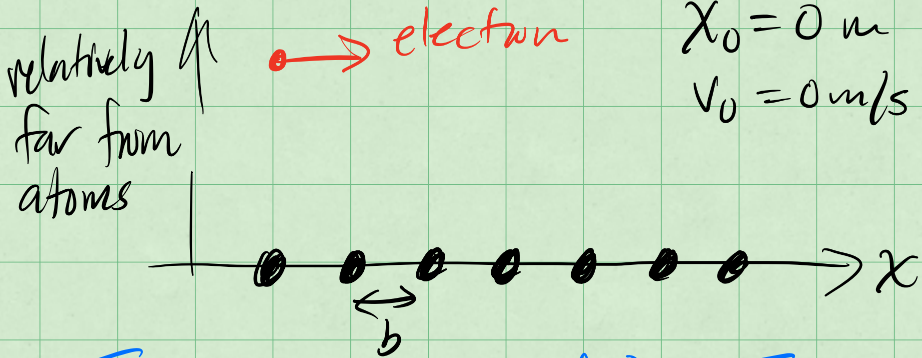
You've likely seen this previously but the existence of a potential is not a given in all circumstances.

We need a "conservative force".

Let's look at a less obvious example

a periodic lattice chain of atoms that can interact with an electron

Example: Lattice Chain



$x_0 = 0\,m$

$v_0 = 0\,m/s$

Force Model $\quad F(x) = -F_0 \sin\left(\frac{2\pi x}{b}\right)$

Work Energy

$$\Delta K = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 = \int_{x_0=0}^{x_1} F(x)\,dx$$

$$\frac{1}{2}mv_1^2 = -\int_0^{x_1} F_0 \sin\left(\frac{2\pi x}{b}\right)dx$$

integral reminder

u sub.

let $u = \frac{2\pi x}{b}$ $\qquad du = \frac{2\pi}{b}dx$

$$\frac{1}{2}mv_1^2 = -\frac{b}{2\pi}F_0 \int_0^{\frac{2\pi x_1}{b}} \sin(u)\,du$$

$$\frac{1}{2}mv_1^2 = -\frac{b}{2\pi}F_0\left[-\cos(u)\right]_0^{\frac{2\pi x_1}{b}}$$

$$\frac{1}{2}mv_1^2 = \frac{b}{2\pi}F_0\left(\cos\left(\frac{2\pi x_1}{b}\right) - 1\right)$$

We can find v_1 given x_1, note there's ambiguity w.r.t. direction,

$$v_1 = \pm\sqrt{\frac{F_0 b}{2\pi}\left(\cos\left(\frac{2\pi x_1}{b}\right) - 1\right)}$$

More importantly note, we can derive a potential,

$$U(x) = -\int F(x)\,dx = \frac{b}{2\pi}F_0\cos\left(\frac{2\pi x}{b}\right)$$

Again, $K + U = \text{const}$ just very different U's.

Conditions for a conservative force

Each condition can define the others as they are mathematically related.

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(1) if $\int_C \vec{F} \cdot d\vec{r}$ is path independent

(2) if $\oint \vec{F} \cdot d\vec{r} = 0$

(3) if $\nabla \times \vec{F} = 0$

Why do these imply eachother?

if $\nabla \times \vec{F} = 0$ then,

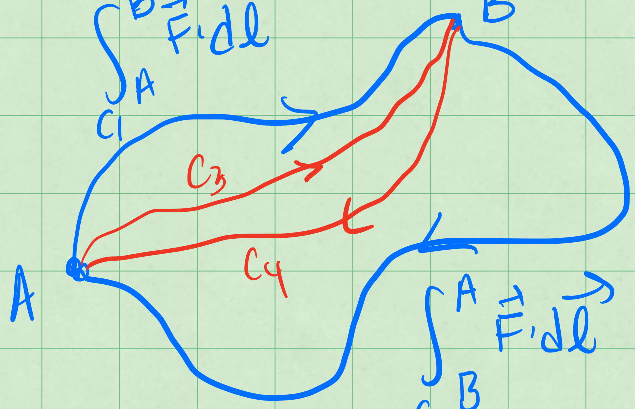
$$\underbrace{\int_S (\nabla \times \vec{F}) \cdot d\vec{A}}_{\text{integral of curl of fnc. over a surface}} = 0 = \underbrace{\oint_C \vec{F} \cdot d\vec{l}}_{\text{intergral of function along edge.}}$$

for any S → Stoke's theorem. ← for the resulting C

↓

if $\oint_C \vec{F} \cdot d\vec{l} = 0$ for any C then

$\int_{A\,(C_1)}^{B} \vec{F}\cdot d\vec{l}$



$\int_{B\,(C_2)}^{A} \vec{F}\cdot d\vec{l}$

$$\oint_C = \int_{A\,(C_1)}^{B} \vec{F}\cdot d\vec{l} + \int_{B\,(C_2)}^{A} \vec{F}\cdot d\vec{l} = \int_{A\,(C_3)}^{B} \vec{F}\cdot d\vec{l} + \int_{B\,(C_4)}^{A} \vec{F}\cdot d\vec{l}$$

must be path independent ⟸ equal size opp sign

again, equal size opp sign

so,

$$\int_A^B \vec{F}\cdot d\vec{l} = -\int_B^A \vec{F}\cdot d\vec{l} \quad \text{for any } C$$

means $\oint_C \vec{F}\cdot d\vec{l} = 0$ again

Summary of Energy Results

$$E = \underbrace{T+V}_{} = \underbrace{K+U}_{}$$

both used to describe KE + PE

Conservation of energy

$$\frac{dE}{dt} = 0$$

when all the forces are conservative

$\Rightarrow$ energy is conserved (really mechanical energy + PE)

Conservative forces

(i) $\vec{F} = \vec{F}(\vec{r})$ function of position only

(ii) $\nabla \times \vec{F} = 0$

that is, $$\left(\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y}\right)\hat{i} + \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0$$

(iii) $\vec{F} = -\nabla U$ the force is the negative gradient of U 16

$$\vec{F} = \left\langle -\frac{dU}{dx}, -\frac{dU}{dy}, -\frac{dU}{dz} \right\rangle$$

(iv) $\displaystyle\int_A^B \vec{F}\cdot d\vec{r}$ is path independent

$$\underbrace{U(B) - U(A)}_{\text{potential energy difference, } \Delta U} = -\int_A^B \vec{F}\cdot d\vec{r}$$

What about conservation of $\vec{p}$ & $\vec{L}$?

$$\frac{d\vec{p}_{sys}}{dt} = \vec{F}_{EXT} \qquad \frac{d\vec{L}_{sys}}{dt} = \vec{\tau}_{Ext}$$

Let's start with conservation of $\vec{p}$.

Conservation of linear momentum 17

$$\vec{F}_{total} = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_i$$

all forces on a system

for a given object, i, there are internal and external forces,

$$\vec{F}_i = \vec{F}_i^{int} + \vec{F}_i^{ext}$$

the internal forces on i are from interactions with the N other objects

$$\vec{F}_i^{int} = \sum_{j \neq i}^{N} \vec{F}_{ij}$$

force of j on i   note $j \neq i$ no self interactions

For example (solar system)

$$\vec{F}_{ij} = -\frac{G M_i M_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j)$$

Consider a 2 body system $(N=2)$ 18

$$\sum_{i=1}^{N=2} \vec{F}_i^{\,int} = \sum_{i=1}^{N=2} \sum_{j \neq i}^{N=2} \vec{F}_{ij} = \vec{F}_{12} + \vec{F}_{21} = 0$$

Newton 3^{rd} $\quad \vec{F}_{12} = -\vec{F}_{21}$

Cool, what about $N=3$?

$$\sum_{i=1}^{N=3} \vec{F}_i^{\,int} = \sum_{i=1}^{N=3} \sum_{j \neq i}^{N=3} \vec{F}_{ij}$$

$$= \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{31} + \vec{F}_{32}$$

$$= \underbrace{(\vec{F}_{12} + \vec{F}_{21})}_{\overset{0}{N3}} + \underbrace{(\vec{F}_{13} + \vec{F}_{31})}_{\overset{0}{N3}} + \underbrace{(\vec{F}_{23} + \vec{F}_{32})}_{\overset{0}{N3}} = 0$$

hmm...

Let's try this in general for N objects,

$$\sum_{i=1}^{N} \vec{F}_i^{\,int} = \sum_{i=1}^{N} \sum_{j \neq i}^{N} \vec{F}_{ij}$$

this sum is equivalent to picking one object at a time and not double counting,

$$\sum_{i=1}^{N}\sum_{j\neq i}^{N}\vec{F}_{ij} = \sum_{i=1}^{N}\sum_{j>i}^{N}\underbrace{\left(\vec{F}_{ij}+\vec{F}_{ji}\right)}_{0 \text{ always}}$$

$$\sum_{i=1}^{N}\vec{F}_i^{\,int} = 0$$

internal forces have 3rd law pairs

Let's look back at the momentum,

$$\vec{P}_{sys} = \sum_{i=1}^{N} m_i\vec{v}_i = \sum_{i=1}^{N}\vec{p}_i$$

$$\frac{d\vec{P}_{sys}}{dt} = \sum_{i=1}^{N} m_i\frac{d\vec{v}_i}{dt} = \sum_{i=1}^{N} m_i\vec{a}_i = \sum_{i=1}^{N}\underbrace{\vec{F}_i^{\,net}}$$

$$\vec{F}_i^{\,net} = \vec{F}_i^{\,ext} + \vec{F}_i^{\,int}$$

$\Leftarrow$ net force on ith object

if there are no external forces,

$$\vec{F}_i^{net} = \vec{F}_i^{int}$$

$$\frac{d\vec{P}_{sys}}{dt} = \sum_{i=1}^{N} \vec{F}_i^{int} = 0$$

(no change in the system momentum.)

So if the system has no external forces $d\vec{P}_{sys}/dt = 0$

and $\Delta \vec{P}_{sys} = 0$ or $\vec{P}_{sys,f} = \vec{P}_{sys,i}$

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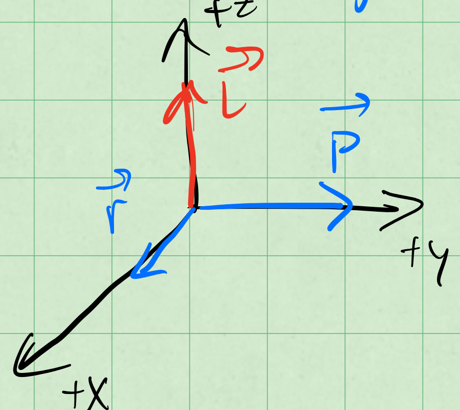
finally, let's start discussing angular momentum, $\vec{L}_{sys}$. fundamentally 3D

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = m(\vec{r} \times \vec{v})$$



When is $d\vec{L}_{sys}/dt = 0$?

HW 1 Ex 4 $\Rightarrow \frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(m(\vec{r} \times \vec{v})\right) = m\frac{d\vec{r}}{dt} \times v + m\vec{r} \times \frac{d\vec{v}}{dt}$$

$\vec{v} \times \vec{v} = 0$

b/c same vec.

$$\frac{d\vec{L}}{dt} = m\vec{r} \times \frac{d\vec{v}}{dt} = \vec{r} \times (m\, d\vec{v}/dt) = \vec{r} \times m\vec{a}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

if $\vec{r} \parallel \vec{F}$

then $\vec{r} \times \vec{F} = 0$

and $\vec{L}$ conserved

$$\vec{\tau} = \vec{r} \times \vec{F}$$

no torques?

What form might we expect for $\Delta \vec{L} = 0$ when there are no external forces?

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Let $\vec{F}_i = \vec{F}_i^{ext} + \vec{F}_i^{int}$

0

each object

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i \qquad \frac{d\vec{l}_i}{dt} = \vec{r}_i \times \vec{F}_i$$

individual ang. mom.

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} \frac{d\vec{l}_i}{dt} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i$$

$$\vec{F}_i = \sum_{j \neq i}^{N} \vec{F}_{ij}$$ ⇐ from before → all internal forces

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} \vec{r}_i \times \left(\sum_{j \neq i}^{N} \vec{F}_{ij} \right)$$

prior result

$$= \sum_{i=1}^{N} \sum_{j>i}^{N} \left(\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji} \right)$$

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

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$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{N} \sum_{j>i}^{N} (\vec{r}_i - \vec{r}_j) \times (\vec{F}_{ij})$$

if $\underbrace{\vec{F}_{ij} \parallel \text{ to } \vec{r}_i - \vec{r}_j}_{\text{gravitational force?}}$ then $\frac{d\vec{L}}{dt} = 0$