CW2 - Making Classical Models ①

The central enterprise of physics is making and testing models of physical systems.

In Classical Mechanics, these models are typically some "equation of motion" [EOM].

An equation of motion describes the evolution of the agents (particles) as the interact with their surroundings & each other.

Typically, our EOMs are ordinary differential equations arising from the model of the interactions.

Examples you have seen

From a Newtonian perspective, we have,

$$\vec{F}_{net} = m\vec{a} = m\ddot{\vec{x}} \quad \leftarrow \quad \ddot{x} = \frac{d^2x}{dt^2}$$

such that,

$$\frac{d^2\vec{x}}{dt^2} = \frac{\vec{F}_{net}}{m}$$

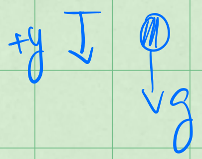
is the general EOM that describes the dynamics of the particle of mass, m.

Dynamics - the (typically) time evolution of the system in question

Specific Examples

1D cases → falling ball (only g); spring-mass

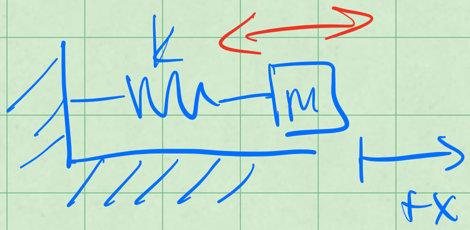
Falling ball (no drag)



$$F_{net,y} = W = mg = m\ddot{y}$$

$\ddot{y} = g$ is the EOM of the ball

Spring-Mass

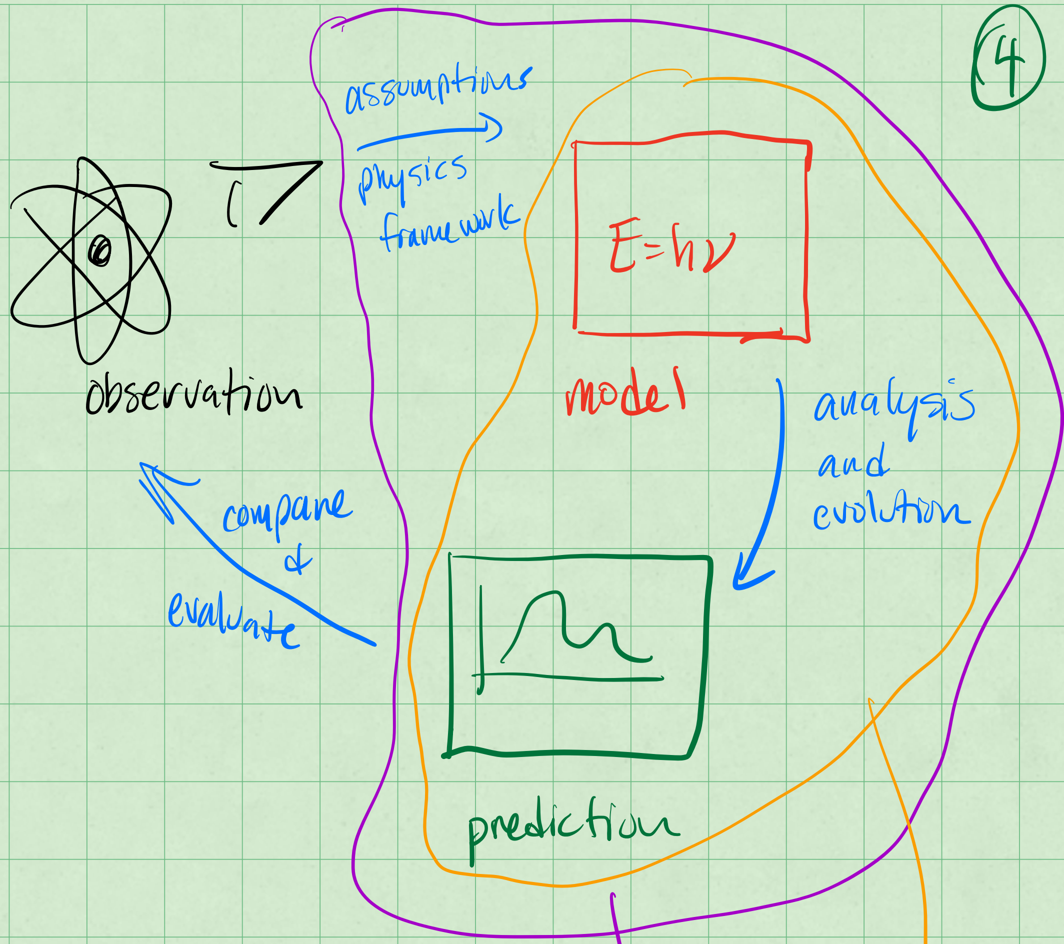


$$F_{net,x} = -kx = m\ddot{x}$$

$\ddot{x} = \frac{-k}{m}x$ is the EOM of the block

But how do we get to these EOMs from a particular situation?

Let's introduce a schematic to make sense of what we are doing.



most of our class focuses on these elements

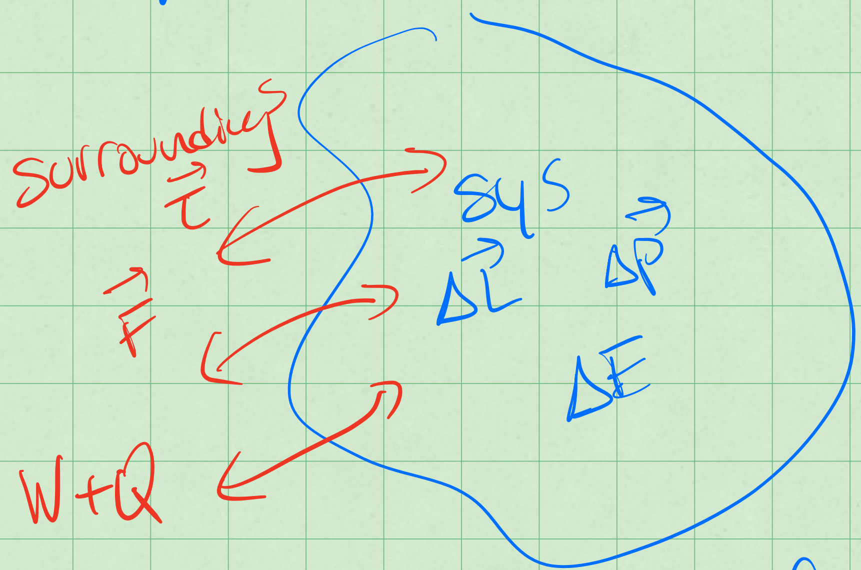
this will be most of the work you do in this class.

Great, but how do we consider these elements?

<u>Practice</u> & <u>discussion</u>

Making models in this class is greatly helped by:

- Identifying the phenomenon & system of interest.
- Identifying the interactions the system has with its surroundings

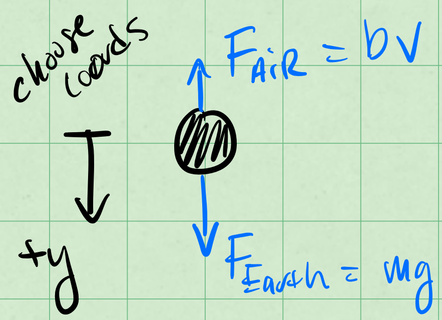


- Choose an appropriate physics framework to investigate your system
  (Newton? Lagrange? Continuous? Discrete?)

- Sketch the system, identify the interactions, name them
- choose your coordinate system
- apply the physics framework

$\Rightarrow$ obtain EOMs $\longrightarrow$ predict

Example: Falling Ball



] model choice

Framework?

Newton to start.

$$F_{net,y} = mg - bv = ma_y$$

EOM: $m\ddot{y} = mg - bv$

$$\ddot{y} = g - \frac{b}{m} v$$

Question: What happens when $\ddot{y} = 0$?

$$\ddot{y} = g - \frac{b}{m}v = 0 \qquad \boxed{v_{term} = \frac{mg}{b}} \; ?$$

terminal velocity for linear drag

Question: can we solve this?

yes! but later

$$\ddot{} = g - \frac{b}{m}\dot{x} \Rightarrow \dot{v} = g - \frac{b}{m}v$$

For now let's hack off the drag bit,

$$\ddot{y} = g \quad \text{our simplified EOM}$$

$$\frac{d^2y}{dt^2} = g \quad \text{or} \quad \frac{dv}{dt} = g \quad \text{and} \quad \frac{dy}{dt} = v$$

2nd order ODE $\qquad$ 2 1st order ODEs

this is one of our first techiques for dealing with ODEs.

$$\frac{dv}{dt} = g \Rightarrow \text{a constant}$$

$$\int_{v_0}^{v(t)} dv = \int_0^t g\,dt \Rightarrow v(t) - v_0 = gt$$

$$\boxed{v(t) = v_0 + gt}$$ constant accel

$$\frac{dy}{dt} = v = v_0 + gt$$

$$\int_{y_0}^{y(t)} dy = \int_0^t (v_0 + gt)\,dt$$

$$y(t) - y_0 = v_0 t + \frac{1}{2}gt^2$$

$$\boxed{y(t) = y_0 + v_0 t + \frac{1}{2}gt^2}$$ constant accel

why plus?

+y $\downarrow$ $\quad \downarrow F_{EARTH}$

Awesome! But what if we weren't sure we could integrate these EOMs directly?

Enter the discrete form:

In 1D,

$$\frac{d^2y}{dt^2} = \frac{F_{net}}{m} \Rightarrow \frac{dv}{dt} = \frac{F_{net}}{m} \quad \& \quad \frac{dy}{dt} = v$$

$$\frac{\Delta v}{\Delta t} = \frac{F_{net}}{m} \qquad \frac{\Delta y}{\Delta t} = v$$

small Δt,

$$v(t+\Delta t) = v(t) + F(t)\Delta t/m$$

velocity update "Euler Step"

Given information @ time t, $F(t)$ & $v(t)$ we can predict $v(t+\Delta t)$ with Δt small.

$$\boxed{v(t+\Delta t) = v(t) + \left(F(t)/m\right)\Delta t}$$

But that's just the velocity. How can we find $y(t+\Delta t)$?

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$$\frac{dy}{dt} = v \Rightarrow \frac{\Delta y}{\Delta t} = \underline{\underline{v_{average}}}$$

$$y(t+\Delta t) = y(t) + \underbrace{v_{avg}}_{\text{What goes here?}} \Delta t$$

$v(t)$?

$v(t+\Delta t)$?

$\frac{v(t+\Delta t)+v(t)}{2}$?

It turns out the best choice (with a small Δt) for now is $v(t+\Delta t)$ the value we predicted earlier

$$\boxed{y(t+\Delta t) = y(t) + v(t+\Delta t)\,\Delta t}$$

Taken together, we have developed a simple numerical integrator.

The Euler-Cromer Step.

In 3D

$$\boxed{\begin{aligned}\vec{v}(t+\Delta t) &= \vec{v}(t) + \frac{\vec{F}(t)}{m}\Delta t \\ \vec{r}(t+\Delta t) &= \vec{r}(t) + v(t+\Delta t)\,\Delta t\end{aligned}}$$

CW3 - Forces & Motion w/ Newton

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The modeling work that we do in classical mechanics leads us to EOMs. These EOMs can be investigated in a number of ways: (1) finding trajectories ⇝ $x(t)$, $v(t)$

later → $x(v)$ & $x(p)$

(phase trajectories)

(2) creating phase space diagrams

$x(v)$ or $x(p)$ for a region of x & v (or p)

(3) fixed points and stability analyses

$\dot{\vec{x}} = 0$ gives $\vec{x}_*$ critical pts.

and so on....

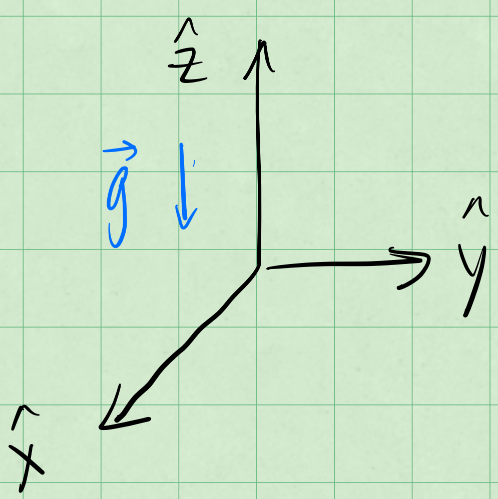
We will start with the:

FBD → EOM → trajectory pipeline

Throughout the analyses that we do, we will ask conceptual questions about these systems.

While doing that we will try also to make clear a number of processes that help us make sense of new models.

Example: Falling Object



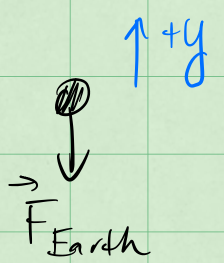
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\vec{F} = m\vec{g}$

In this coordinate system,

$\vec{F} = -mg\hat{y}$

Focus on the 1D problem



$$\vec{F}_{earth} = -mg\hat{y} = m\vec{a} = m\ddot{\vec{r}}$$

$$\ddot{\vec{r}} = -g\hat{y} \qquad \ddot{y} = \dot{v} = -g$$

$$dv = -g\,dt$$

$$v(t) - v_0 = \int_{v_0}^{v(t)} dv = -g\int_{t_0}^{t} dt' = -g(t - t_0)$$

(3)

$$\boxed{v(t) = v_0 - g(t - t_0)} \quad \text{trajectory of } v$$

Integrate again to find $y(t)$,

$$\boxed{y(t) = y_0 + v_0 t - \tfrac{1}{2} g t^2 \quad \text{trajectory of } y}$$

Let's add drag to the model

Drag is the result of collisions with the falling body. Drag models are empirically developed as the specifics of how those collisions impact the moving body are quite complex. The two simplest models we have for drag are:

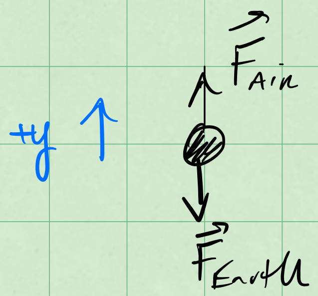
Linear Drag: $F_{lin} = \gamma v \rightarrow$ small & slow

Quadratic Drag: $F_{quad} = D v^2$ large & fast

Both point opposite the velocity, $-\hat{v} = \dfrac{-\vec{v}}{|v|}$

Critically: you need to consider your coordinate system

Example: Quadratic Drag (1D) ④



$$\vec F_{net} = \vec F_{AIR} + \vec F_{EARTH} = m\vec a = m\ddot{\vec r}$$

$$F_{Air} = Dv^2 \qquad F_{EARTH} = mg$$

$$m\ddot y = -mg + Dv^2$$

EOM: $$\boxed{\ddot y = -g + \frac{D}{m} v^2}$$

let $\tilde D = D/m$

How do we solve this EOM?

- separation of variables, we try to right each side as functions only of v and t

$$\int f(v)\,dv = \int g(t)\,dt$$

If we can integrate them we will get a closed form relationship between v and t.

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$$\ddot{y} = -g + \tilde{D}v^2 \Longrightarrow \dot{v} = \frac{dv}{dt} = -g + \tilde{D}v^2$$

$$\frac{dv}{dt} = -g + \tilde{D}v^2$$

$$\frac{dv}{-g+\tilde{D}v^2} = dt$$

$$\frac{dv}{dt} = 0 = -g + \tilde{D}v^2 \Rightarrow v_t = \sqrt{g/\tilde{D}}$$

terminal velocity

$$\frac{dv}{g\left[\left(v/v_t\right)^2 - 1\right]} = dt$$

so $f(v) = \dfrac{1}{\left(\frac{v}{v_T}\right)^2 - 1}$

$g(t) = g$

$$\frac{dv}{\left(\frac{v}{v_t}\right)^2 - 1} = g\,dt$$

Drop from rest

$$\int_0^{v(t)} \frac{dv'}{\left(\frac{v'}{v_t}\right)^2 - 1} = \int_0^t g\,dt'$$

a trajectory requires initial conditions

$v(0) = 0$

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$$\int_0^v \frac{dv'}{\left(\frac{v'}{v_t}\right)^2 - 1} = \int_0^t g\,dt$$

→ look up integrals

⇒ WolframAlpha, sympy, etc.

$$-v_t \tanh^{-1}\left(\frac{v}{v_t}\right) = gt$$

$$\boxed{v(t) = v_t \tanh\left(\frac{-gt}{v_t}\right)}$$

where $v_t = \sqrt{g/\tilde{D}} = \sqrt{mg/D}$

New equation! How do we check it?

⇒ units? ⇒ limits?