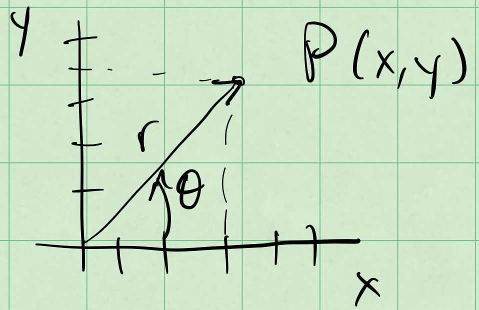
Vectors

in 2D



$$x^2+y^2=r^2 \qquad x=r\cos\theta \qquad y=r\sin\theta$$

Plane Polar coordinates (r,θ)

$$\vec{r} = x\hat{x} + y\hat{y} = x\hat{e}_x + y\hat{e}_y = x\hat{i} + y\hat{j}$$

unit vectors – for Cartesian, fixed in space/time

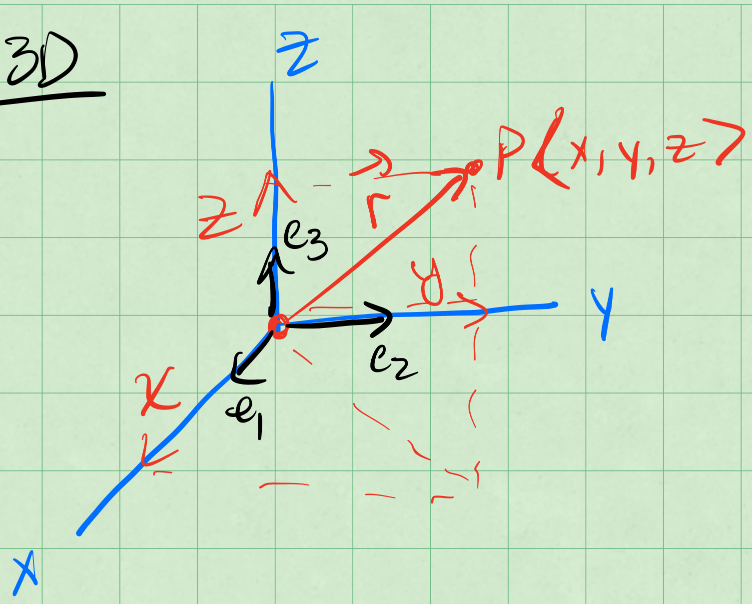
Claim $\vec{r} = |\vec{r}|\hat{r}$ no $\hat{\theta}$

$$\vec{r} = x\hat{x} + y\hat{y} \qquad |\vec{r}| = \sqrt{x^2+y^2}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$$

$$\vec{r} = |\vec{r}|\hat{r} = x\hat{x} + y\hat{y} \quad \checkmark$$

In 3D



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 \text{ etc.}$$

Unit Vectors

Cartesian unit vectors are fixed in space/time in inertial frames.

magnitude $= 1$ $\quad |\hat{i}| = 1 \quad |e_2| = 1$ etc.

The are orthogonal $\Rightarrow$ their dot product vanishes b/c they are $\perp$

$\hat{x} \cdot \hat{y} = 0 \qquad \hat{z} \cdot \hat{z} = 1$

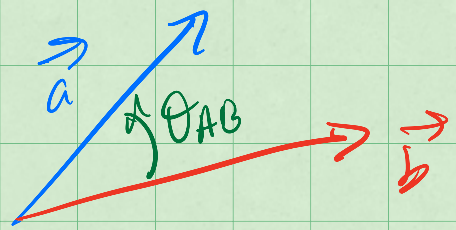
$\hat{e}_1 \cdot \hat{e}_3 = 0 \qquad \hat{e}_2 \cdot \hat{e}_2 = 1$

etc. $\qquad$ etc...

Dot Products (Inner Products)

$$\vec{a} \cdot \vec{b} = \langle a_x, a_y, a_z \rangle \cdot \langle b_x, b_y, b_z \rangle$$

$$= a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos\theta_{AB}$$



The dot product is distributive,

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

PROOF:

$$\vec{a} \cdot [\vec{b} + \vec{c}] = \langle a_x, a_y, a_z \rangle \cdot \langle b_x + c_x, b_y + c_y, b_z + c_z \rangle$$

$$= a_x(b_x + c_x) + a_y(b_y + c_y) + a_z(b_z + c_z)$$

$$= (a_x b_x + a_x c_x) + (a_y b_y + a_y c_y) + (a_z b_z + a_z c_z)$$

$$= (a_x b_x + a_y b_y + a_z b_z) + (a_x c_x + a_y c_y + a_z c_z)$$

$$= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \vec{a}\cdot(\vec{b}+\vec{c}) \quad \checkmark$$

Cross ("vector") product

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

cross multiply to find component

$$= \hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x)$$

$$+ \hat{k}(a_x b_y - a_y b_x)$$

Note: there's a sign change here

$$\vec{a}\times\vec{b} =$$

$$\langle a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ \underbrace{a_x b_y - a_y b_x}\rangle$$

no x comp. no y comp note: no z component a few notes about cross products, 17

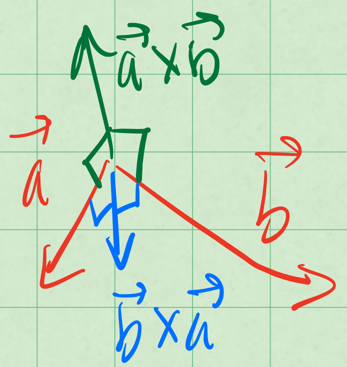
1) $\vec{a} \times \vec{b}$ always produces a vector never a scalar

2) $(\vec{a} \times \vec{b})_i$ denotes the ith component of $\vec{a} \times \vec{b}$; a scalar

3) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ order matters

Question: what is $\vec{a} \times \vec{b}$ relation to $\vec{b} \times \vec{a}$?

RH Rule: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$



Units Reminder

18

Truly: units are helpful.... very much so.

$$[\vec{r}] = \text{length}$$

$$[\vec{v}] = \text{length}/\text{time}$$

$$[\vec{a}] = \text{length}/\text{time}^2$$

$$[\vec{F}] = \frac{(\text{mass})(\text{length})}{(\text{time})^2}$$

$$[\vec{p}] = \frac{(\text{mass})(\text{length})}{\text{time}}$$

etc....

$$[E] = \frac{(\text{mass})(\text{length})^2}{\text{time}^2}$$

Let's Revisit our Drag Model

$$F(v) = cv + dv^2 + O(v^3)$$

Question: What are the units of the drag coefficients? 19

$$[F] = \frac{(\text{mass})(\text{length})}{\text{time}^2}$$

$$[\alpha_n v^n] = [\alpha_n]\left(\frac{\text{length}}{\text{time}}\right)^n$$

all coeffs

$a_1 = c$

$a_2 = d$

etc...

$$[F] = [\alpha_n][v^n]$$

$$[a_n] = [F]/[v^n]$$

$$= \frac{(\text{mass})(\text{length})}{(\text{time})^2}\left(\frac{\text{time}}{\text{length}}\right)^n$$

$$[a_n] = (\text{mass})(\text{time})^{n-2}(\text{length})^{1-n}$$

$$= \frac{(\text{mass})(\text{time})^{n-2}}{(\text{length})^{n-1}}$$

Check:

$$[a_1] = [c] = \frac{(\text{mass})(\text{time})^{-1}}{(\text{length})^0} = \frac{\text{mass}}{\text{time}} \left(\frac{\text{length}}{\text{time}} \right)$$

20

$\checkmark$ checks out

$$[a_2] = [d] = \frac{(\text{mass})(\text{time})^0}{(\text{length})^1} = \frac{\text{mass}}{\text{length}} \left(\frac{\text{length}^2}{\text{time}^2} \right)$$

v^2

also checks out