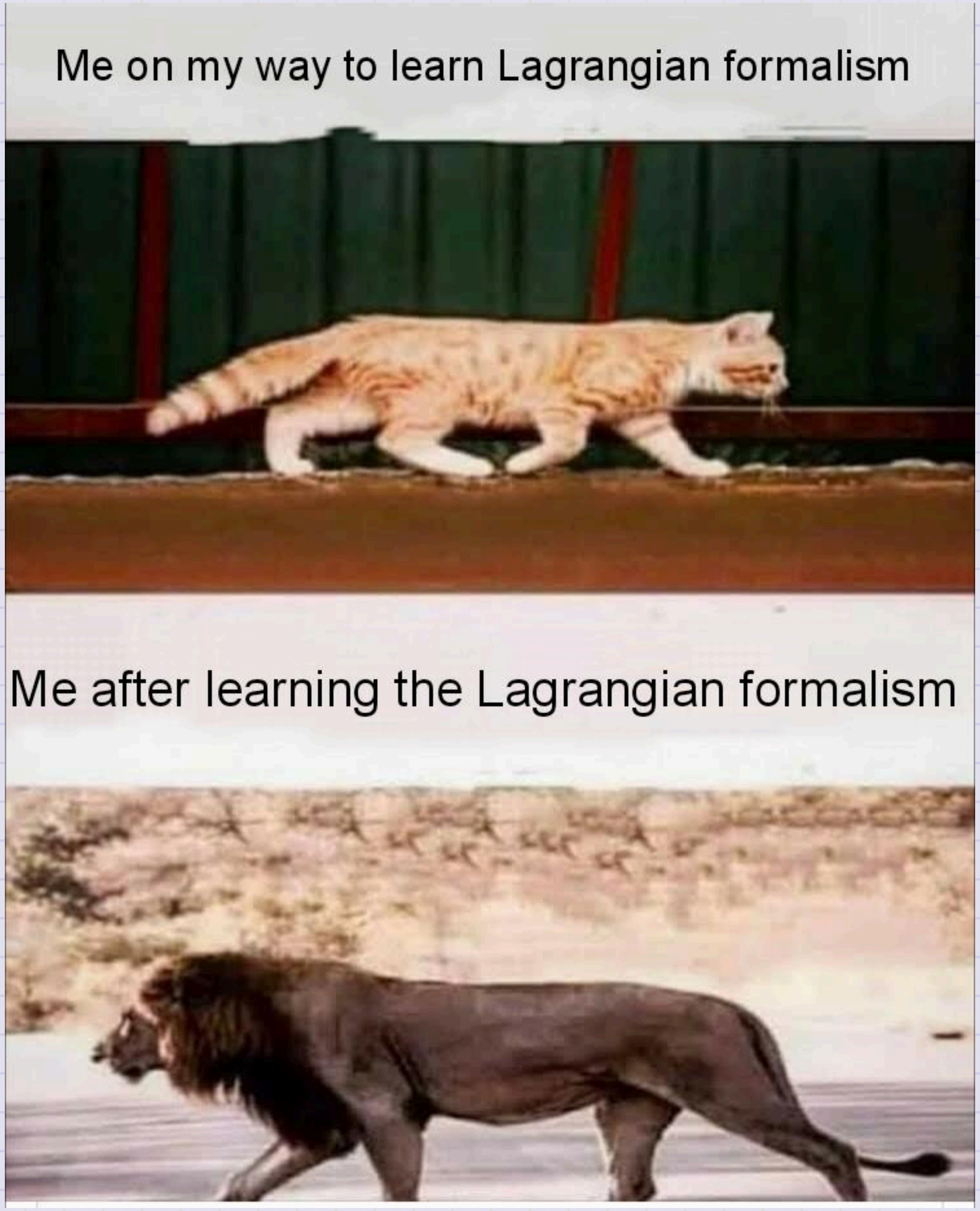
Day 33 - Lagrangian Examples



Announcements

Assignments

- Homework 8 is posted (Last HW; Due Nov 21)
- Rubric for final project is posted

Announcements

Rest of Semester Schedule

- Week 12 - Intro to Lagrangian Dynamics
- Week 13 - Examples of Lagrangian Dynamics
- Week 14 - Project Prep (Thanksgiving week)
- Week 15 - Presentations (Last week of class)
- Week 16 - Computational Essay Due (Monday of Finals week)

NO IN-CLASS FINAL EXAM

Announcements

- **Friday (11/14) Class**: DC will be in classroom at 11:30a
  - Hosting speaker @ 12:30p
  - Classroom open from 11:30a-12:50p
  - Second Midterm Help Session
- **Friday (11/14) Office Hours**: 2:00-3:00p (shortened due to committee meeting)

Quick question

A former PHY 321 student would like to survey you about generative AI as he is helping to develop guidelines for MSU. Would you be willing to fill out a quick survey next week?

1. Sure
2. No thanks
3. Maybe

FWIW, it's not going to take more than 5-10 minutes.

Reminder: The Lagrangian

The Lagrangian $\mathcal{L}$ is a function that summarizes the dynamics of the system. It is typically defined as:

$$\mathcal{L}(q, \dot{q}, t) = T - V$$

where:

- T is the **kinetic energy** of the system (depends on **gen. vel.**, $\dot{q}$),
- V is the **potential energy** of the system (depends on **gen. pos.**, q).

The equation of motion is recovered by applying the Euler-Lagrange equation to the Lagrangian (minimizing the action integral).

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Clicker Question 33-1

For a 1D SHO, the kinetic and potential energy are given by:

$$T = \frac{1}{2}m\dot{x}^2 \quad \text{and} \quad V = \frac{1}{2}kx^2$$

What are the derivatives of the Lagrangian $\mathcal{L} = T - V$ with respect to x and $\dot{x}$?

1. $\frac{\partial \mathcal{L}}{\partial x} = kx$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$
2. $\frac{\partial \mathcal{L}}{\partial x} = -kx$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$
3. $\frac{\partial \mathcal{L}}{\partial x} = kx$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}} = -m\dot{x}$
4. $\frac{\partial \mathcal{L}}{\partial x} = -kx$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}} = -m\dot{x}$
5. None of the above.

Clicker Question 33-2

For the plane pendulum, with $\mathcal{L}(x, \dot{x}, y, \dot{y}, t) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$

We found:

$$\frac{d}{dt}(m\dot{x}) = 0 \qquad \text{and} \quad \ddot{y} = -g$$

Does that seem right?

1. Yes, it's fine.
2. Maybe, but I'm not sure I can tell you why.
3. No, I know this is wrong, but I'm not sure why.
4. No, this is definitely wrong and I can prove it!

Clicker Question 33-3

For the plane pendulum, we changed the Lagrangian from Cartesian coordinates to plane polar coordinates. In Cartesian, we found the Lagrangian depended on $y, \dot{x}, \dot{y}$. In polar, it only depended on ϕ and $\dot{\phi}$.

$$\mathcal{L}(x, y, \dot{y}) \longrightarrow \mathcal{L}(\phi, \dot{\phi})$$

What does that tell you about the dimensions of the system? The system is:

1. in 3D space, so it's 3D.
2. described by two spatial dimensions (x, y), so it's 2D.
3. described by one spatial dimension (ϕ), so it's 1D.

We chose our generalized coordinates poorly

We used the Lagrangian formalism to derive the equations of motion for a plane pendulum. We chose the x and y coordinates.

$$T(\dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad V(y) = mgy$$

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

This gave us the following derivatives for the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = -mg \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = -m\ddot{y}$$

We made a mistake by not including the constraint

We made a mistake by not including the constraint $x^2 + y^2 = L^2$ in our Lagrangian.

We can change variables to r and ϕ.

$$x = r\cos(\phi) \quad y = r\sin(\phi)$$

$$T(\dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(r^2\dot{\phi}^2 + 2r\dot{r}\dot{\phi} + \dot{r}^2\right) = T(r, \dot{r}, \phi, \dot{\phi})$$

$$V(y) = mgy = mgr\sin(\phi) = V(r, \phi)$$

Now we include the constraint $r = L$, so that $\dot{r} = 0$.

$$T(\phi, \dot{\phi}) = \frac{1}{2}mL^2\dot{\phi}^2 \quad V(\phi) = mgL\cos(\phi)$$

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\phi}^2 - mgL\cos(\phi)$$

Clicker Question 33-4

For the plane pendulum, we changed the Lagrangian from Cartesian coordinates to plane polar coordinates. In Cartesian, we found the Lagrangian depended on $y, \dot{x}, \dot{y}$. In polar, it only depended on ϕ and $\dot{\phi}$.

$$\mathcal{L}(x, y, \dot{y}) \longrightarrow \mathcal{L}(\phi, \dot{\phi})$$

What does that tell you about the dimensions of the system? The system is:

1. in 3D space, so it's 3D.
2. described by two spatial dimensions (x, y), so it's 2D.
3. described by one spatial dimension (ϕ), so it's 1D.

Clicker Question 33-5

With $\mathcal{L} = \dfrac{1}{2} m L^2 \dot{\phi}^2 - mgL\cos(\phi)$, we can find the equations of motion.

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

Which of the following equations of motion is correct?

1. $\ddot{\phi} = -\frac{g}{L}\sin(\phi)$
2. $\ddot{\phi} = -\frac{g}{L}\cos(\phi)$
3. $\ddot{\phi} = -\sqrt{\frac{g}{L}\sin(\phi)}$
4. $\ddot{\phi} = -\sqrt{\frac{g}{L}\cos(\phi)}$
5. None of these

Clicker Question 33-6

For the Atwood's machine, M is connected to m by a string of length l. Each mass has a length of string extended as measured from the center of the pulley (R) of y_1 and y_2, respectively. The string wraps around half the pulley.

Which of the following represents the equation of constraint for the system?

1. $y_1 + y_2 = l - R\phi$
2. $y_1 - y_2 = l + R\phi$
3. $y_1 + y_2 = l - \pi R$
4. $y_1 - y_2 = l + \pi R$
5. None of these

Take the time derivative of the constraint equation. What do you notice?

Clicker Question 33-7

With a Lagrangian of the form $\mathcal{L} = \frac{1}{2}(M+m)\dot{y}_1^2 - (M-m)gy_1$, we can find the **generalized forces** and **generalized momenta**.

$$F_{y_1} = \frac{\partial \mathcal{L}}{\partial y_1} = -\frac{\partial V}{\partial y_1} \quad p_{y_1} = \frac{\partial \mathcal{L}}{\partial \dot{y}_1} = \frac{\partial T}{\partial \dot{y}_1}$$

What are F_{y_1} and p_{y_1} for the Atwood's machine?

1. $F_{y_1} = -mg$ and $p_{y_1} = m\dot{y}_1$
2. $F_{y_1} = -Mgy_1$ and $p_{y_1} = M\dot{y}_1$
3. $F_{y_1} = -(M-m)g$ and $p_{y_1} = (M+m)\dot{y}_1$
4. $F_{y_1} = -(M+m)g$ and $p_{y_1} = (M-m)\dot{y}_1$
5. None of these

Clicker Question 33-8

Now, we allow the pulley (mass, M_p) to rotate. The Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{y}_1^2 + \frac{1}{2}I\dot{\phi}^2 - (M-m)gy_1$$

Where I is the moment of inertia of the pulley. What is the moment of inertia of the pulley?

1. $I = \frac{1}{2}M_pR^2$
2. $I = \frac{1}{3}M_pR^2$
3. $I = M_pR^2$
4. $I = \frac{1}{4}M_pR^2$
5. None of these

Clicker Question 33-9

The rope moves without slipping on the pulley. A rotation of $Rd\phi$ corresponds to a displacement of dy_1 for the first mass, M. What is the **new** equation of constraint for the system?

1. $y_1 + y_2 = l - R\phi$
2. $dy_1 = Rd\phi$
3. $y_1 = R\phi$
4. $\dot{y}_1 = R\dot{\phi}$
5. More than one of these