CW 1 – Intro to Class. Mech. ①

Outline: What is Classical Mechanics?
How do we formulate it?
What are the essential physics models for single particles?
What mathematics do we need to get started?

What is Classical Physics ?

- the study of slow, large things
  - slow? no relativity; no QFT
  - large? no quantum; no stat mech

What about Classical Mechanics ?

- We now add "mechanical" to our conditions and so we exclude electromagnetic systems.

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$\Rightarrow$ not always. we can describe the force on a charged particle using a classical model,

$$\vec{F}_{\text{Lorentz}} = q(\vec{E} + \vec{v} \times \vec{B})$$

How do we formulate Class. Mech.?

We first consider how have seen classical mechanics in the past.

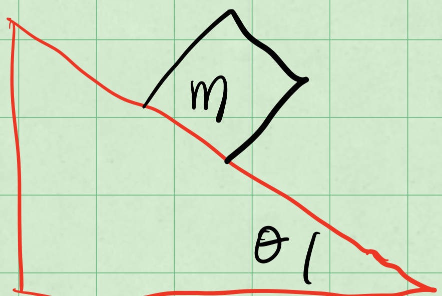
$$\vec{F}_{net} = m\vec{a}$$ Newton's 2nd Law

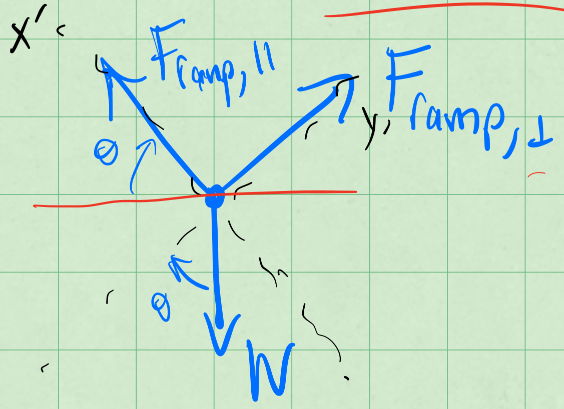
notice this formulation is vector based. That is, the relationship between pushes and accelerations are vectoral. Namely,

$$F_x = ma_x \quad F_y = ma_y \quad F_z = ma_z$$

Each push in a <u>Cartesian</u> <u>direction</u> results in a proportional response - an acceleration <u>in the same direction</u> as the <u>net</u> <u>push</u>.

<u>Ex:</u> Box on a plane with friction. What angle does it slide if coefficient of static friction is μ_s?





→ friction

→ normal

change to x', y'

$\vec{F}_{net} = m\vec{a} = 0$ static

max friction force $= F_{\parallel} = \mu_s F_{\perp}$

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$$\Sigma F_{x'i} = F_{ramp,\parallel} - W\sin\theta = 0$$

$$\Sigma F_{y'i} = F_{ramp,\perp} - W\cos\theta = 0$$

$W = mg$ so that,

$F_{ramp,\parallel} = mg\sin\theta$ $F_{ramp,\perp} = mg\cos\theta$

But $F_{ramp,\parallel,max} = \mu_s mg\cos\theta$

so,

$mg\sin\theta = \mu_s mg\cos\theta$ @ max!

$$\tan\theta = \mu_s$$

$$\theta = \tan^{-1}(\mu_s)$$

$\mu_{steel} = 0.16$ $\sim 9^\circ$

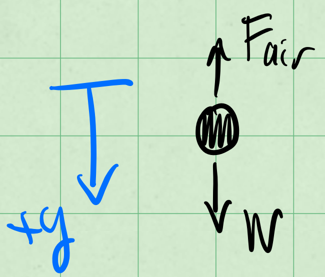
$\mu_{rubber} = 0.8$ $\sim 39^\circ$

Notes: - this was a static problem

$\vec{F}_{net} = 0$

- we rotated the coordinate system to match our ramp
- we still used Cartesian coords.

Ex: Falling Ball in 1D   Predict its motion



models for F_{air} ?

let $F_{air} = F(v)$ just some function of v

In 1D,

$$F_{net,y} = ma_y = +mg - F(v)$$

Assume low v, why?

$\Rightarrow$ Classical Mechanics!

Taylor Expand $F(v)$ & keep low terms ⑥

$$f(x) \simeq \sum_{n=0}^{N} \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=a} (x-a)^n$$

formula for Taylor Expansion around $x=a$.

$$f(x) \simeq f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots$$

Cool. Let's do that for $F(v)$ around $v=0$ when $F_{drag}=0$

$$F(v) = \underbrace{F(0)}_{0} + \underbrace{F'(0)}_{b} v + \underbrace{\frac{F''(0)}{2}}_{c} v^2 + \dots$$

these are just #'s

quadratic drag

$$F(v) \simeq bv + cv^2$$

linear drag

Back to Newton 2,

$$F_y = ma_y = mg - bv - cv^2$$

and thus,

$$a_y = g - \frac{b}{m}v - \frac{c}{m}v^2$$ OOF.

How do we solve this?

$$a_y = g - \frac{b}{m}v - \frac{c}{m}v^2$$

$$\frac{d^2y}{dt^2} = g - \frac{b}{m}\left(\frac{dy}{dt}\right) - \frac{c}{m}\left(\frac{dy}{dt}\right)^2$$

$$\ddot{y} = g - \frac{b}{m}\dot{y} - \frac{c}{m}\dot{y}^2$$

$$\dot{v} = g - \frac{b}{m}v - \frac{c}{m}v^2$$

We will come back to it.

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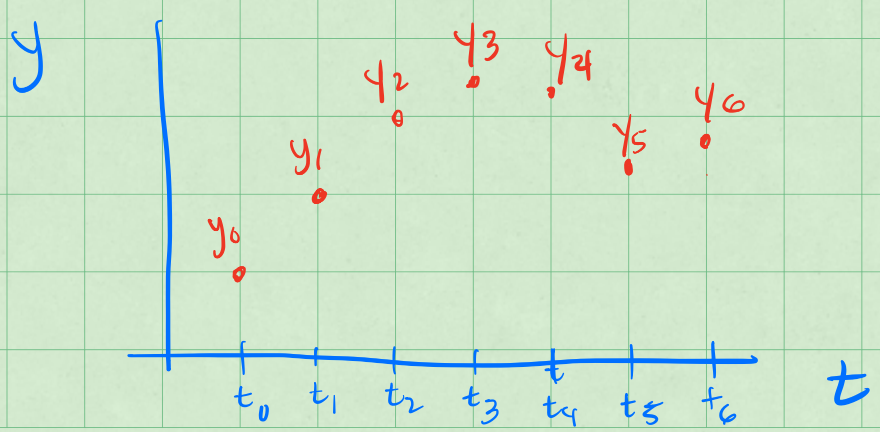
Note: - this is a dynamic 1D problem
- this is a nonlinear problem
- we are stuck @ the moment

Enter Discretization ← another formulation

we posit discrete time, like snapshots of the motion where a given measure of time, t_i exists in a discrete set, from $t_0 \to t_f$
initial → final

$$t \in [t_0, t_f]$$

thus we concieve of a plot of motion as discrete,



t	y
t_0	y_0
t_1	y_1
t_2	y_2
$\vdots$	$\vdots$

if these are equally spaced then,

$$\Delta t = t_{i+1} - t_i \overset{\text{or} \rightarrow}{=} \frac{t_f - t_0}{n}$$

Thus, $t_i = t_0 + i\Delta t$

$y(t_i) = y_i$

Great, but what can we do with this?

Let's define an average velocity over a time step, Δt, like this,

$$\boxed{V(t) = \frac{y(t+\Delta t) - y(t)}{\Delta t}}$$

Avg velocity

$V(t_i) = V_i$ ⟵ discrete V also.

$$\boxed{V_i = \frac{y_{i+1} - y_i}{\Delta t}}$$

Avg velocity (discrete)

Note if we take the limit of $\Delta t \to 0$ (10)
we have the instantaneous velocity,

$$\lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t} = \frac{dy}{dt} = \dot{y}$$

fundamental theorem of calculus

Ok what about the acceleration?

We can also define an average accel over an interval Δt,

$$\boxed{a(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t}}$$

average acceleration

$a(t_i) = a_i$ discrete a,

$$\boxed{a_i = \frac{v_{i+1} - v_i}{\Delta t}}$$

average acceleration (discrete)

Again we can take the limit as $\Delta t \to 0$ to show the instantaneous acceleration

$$\lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \dot{v} = \ddot{y}$$ again FTC

Discrete Formulation of Mechanics

Let there be a 1D net Force, $F(x)$

Here the force changes with location, x, a position dependent force.

$F(x_i) = F_i$ → discretize force.

$a_i = F_i / m$ → Newton 2

$$a_i = \frac{v_{i+1} - v_i}{\Delta t} \Rightarrow v_{i+1} = v_i + a_i \Delta t$$

$$\boxed{v_{i+1} = v_i + \frac{F_i}{m} \Delta t}$$

predict the new velocity just a bit later.

Nice! Now we can predict the new velocity, v_{i+1}, a little time later.

We will pause here and derive these methods for numerical integration later.

The discrete formulation is quite powerful and will help us solve our equations of motion like,

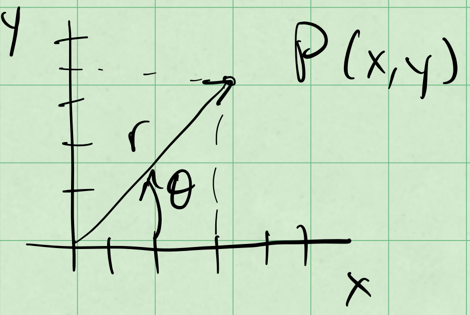
$$a_y = g - \frac{c}{m} v - \frac{d}{m} v^2$$

What mathematical ideas are we going to need? Obviously, algebra & geometry ← lots

- Coordinate Sys & transforms ← lots
- Differential & Integral Calculus ← lots
- Vectors and vector operations ← lots
- Discrete Calculus ← some
- Complex Analysis ← a little

Vectors

in 2D



$$x^2+y^2=r^2 \qquad x=r\cos\theta \qquad y=r\sin\theta$$

Plane Polar coordinates (r,θ)

$$\vec{r} = x\underbrace{\hat{x}} + y\underbrace{\hat{y}} = x\underbrace{\hat{e}_x} + y\underbrace{\hat{e}_y} = x\underbrace{\hat{i}} + y\underbrace{\hat{j}}$$

unit vectors – for Cartesian, fixed in space/time

Claim

$$\vec{r} = |\vec{r}|\,\hat{r}$$

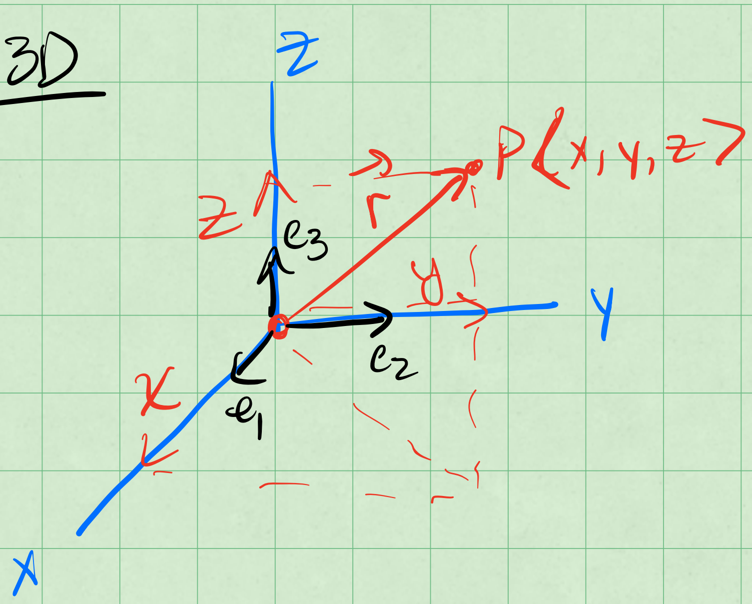
no $\hat{\theta}$

$$\vec{r} = x\hat{x} + y\hat{y} \qquad |\vec{r}| = \sqrt{x^2+y^2}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$$

$$\vec{r} = |\vec{r}|\,\hat{r} = x\hat{x}+y\hat{y} \quad \checkmark$$

In 3D



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 \text{ etc.}$$

Unit Vectors

Cartesian unit vectors are fixed in space/time in inertial frames.

magnitude $= 1$ $\quad |i| = 1 \quad |e_2| = 1$ etc.

The are orthogonal $\Rightarrow$ their dot product vanishes b/c they are $\perp$

$\hat{x} \cdot \hat{y} = 0 \qquad \hat{z} \cdot \hat{z} = 1$

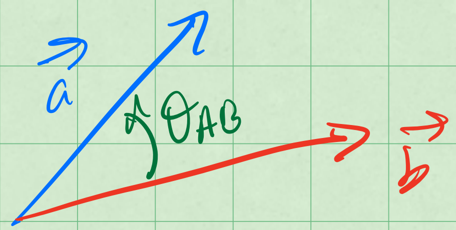
$\hat{e}_1 \cdot \hat{e}_3 = 0 \qquad \hat{e}_2 \cdot \hat{e}_2 = 1$

etc. $\qquad$ etc...

Dot Products (Inner Products)

$$\vec{a}\cdot\vec{b} = \langle a_x, a_y, a_z\rangle \cdot \langle b_x, b_y, b_z\rangle$$

$$= a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}|\cos\theta_{AB}$$



The dot product is distributive,

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

PROOF:

$$\vec{a}\cdot[\vec{b}+\vec{c}] = \langle a_x, a_y, a_z\rangle \cdot \langle b_x+c_x, b_y+c_y, b_z+c_z\rangle$$

$$= a_x(b_x+c_x) + a_y(b_y+c_y) + a_z(b_z+c_z)$$

$$= (a_x b_x + a_x c_x) + (a_y b_y + a_y c_y) + (a_z b_z + a_z c_z)$$

$$= (a_x b_x + a_y b_y + a_z b_z) + (a_x c_x + a_y c_y + a_z c_z)$$

$$= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \vec{a}\cdot(\vec{b}+\vec{c}) \quad \checkmark$$

Cross ("vector") product

$$\vec{a}\times\vec{b} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

cross multiply to find component

$$= \hat{i}(a_y b_z - a_z b_y) - \hat{j}(a_x b_z - a_z b_x)$$
$$+ \hat{k}(a_x b_y - a_y b_x)$$

Note: there's a sign change here

$$\vec{a}\times\vec{b} =$$

$$\langle \underset{\text{no x comp.}}{a_y b_z - a_z b_y},\ \underset{\text{no y comp}}{a_z b_x - a_x b_z},\ \underbrace{a_x b_y - a_y b_x}_{\text{note: no z component}} \rangle$$

a few notes about cross products,

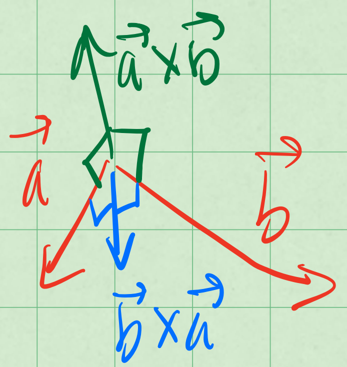
1) $\vec{a} \times \vec{b}$ always produces a vector never a scalar

2) $(\vec{a} \times \vec{b})_i$ denotes the ith component of $\vec{a} \times \vec{b}$; a scalar

3) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ order matters

Question: what is $\vec{a} \times \vec{b}$ relation to $\vec{b} \times \vec{a}$?

RH Rule: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$



Units Reminder

Truly: units are helpful.... very much so.

$$[\vec{r}] = \text{length}$$

$$[\vec{v}] = \text{length}/\text{time}$$

$$[\vec{a}] = \text{length}/\text{time}^2$$

$$[\vec{F}] = \frac{(\text{mass})(\text{length})}{(\text{time})^2}$$

$$[\vec{p}] = \frac{(\text{mass})(\text{length})}{\text{time}}$$

etc....

$$[E] = \frac{(\text{mass})(\text{length})^2}{\text{time}^2}$$

Let's Revisit our Drag Model

$$F(v) = cv + dv^2 + O(v^3)$$

Question: What are the units of the drag coefficients? 19

$$[F] = \frac{(\text{mass})(\text{length})}{\text{time}^2}$$

all coeffs

$a_1 = c$

$a_2 = d$

etc...

$$[\alpha_n v^n] = [\alpha_n]\left(\frac{\text{length}}{\text{time}}\right)^n$$

$$[F] = [\alpha_n][v^n]$$

$$[a_n] = [F]/[v^n]$$

$$= \frac{(\text{mass})(\text{length})}{(\text{time})^2}\left(\frac{\text{time}}{\text{length}}\right)^n$$

$$[a_n] = (\text{mass})(\text{time})^{n-2}(\text{length})^{1-n}$$

$$= \frac{(\text{mass})(\text{time})^{n-2}}{(\text{length})^{n-1}}$$

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Check:

$$[a_1] = [c] = \frac{(\text{mass})(\text{time})^{-1}}{(\text{length})^{0}} = \frac{\text{mass}}{\text{time}} \overbrace{\left(\frac{\text{length}}{\text{time}}\right)}^{v}$$

checks out

$$[a_2] = [d] = \frac{(\text{mass})(\text{time})^{0}}{(\text{length})^{1}} = \frac{\text{mass}}{\text{length}} \overbrace{\left(\frac{\text{length}^2}{\text{time}^2}\right)}^{v^2}$$

also checks out