CW3 – Forces & Motion w/ Newton

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The modeling work that we do in classical mechanics leads us to EOMs. These EOMs can be investigated in a number of ways: (1) finding trajectories ⇝ $x(t)$, $v(t)$

later → $x(v)$ & $x(p)$

(phase trajectories)

(2) creating phase space diagrams

$x(v)$ or $x(p)$ for a region of x & v (or p)

(3) fixed points and stability analyses

$\dot{\vec{x}} = 0$ gives $\vec{x}_*$ critical pts.

and so on....

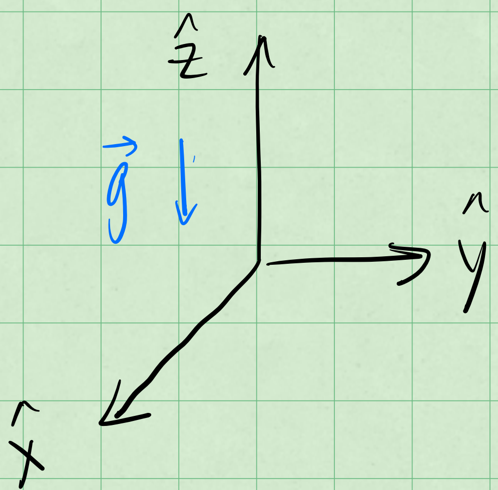
We will start with the:

FBD → EOM → trajectory pipeline

Throughout the analyses that we do, we will ask conceptual questions about these systems.

While doing that we will try also to make clear a number of processes that help us <u>make</u> <u>sense</u> of new models. ②

<u>Example: Falling Object</u>



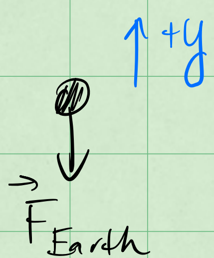
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{F} = m\vec{g}$$

In this coordinate system,

$$\vec{F} = -mg\hat{y}$$

<u>Focus on the 1D problem</u>



$$\vec{F}_{earth} = -mg\hat{y} = m\vec{a} = m\ddot{\vec{r}}$$

$$\ddot{\vec{r}} = -g\hat{y} \qquad \ddot{y} = \dot{v} = -g$$

$$dv = -g\,dt$$

$$v(t) - v_0 = \int_{v_0}^{v(t)} dv = -g\int_{t_0}^{t} dt' = -g(t - t_0)$$

$$\boxed{v(t) = v_0 - g(t - t_0)} \quad \text{trajectory of } v$$ (3)

Integrate again to find $y(t)$,

$$\boxed{y(t) = y_0 + v_0 t - \tfrac{1}{2} g t^2 \quad \text{trajectory of } y}$$

Let's add drag to the model

Drag is the result of collisions with the falling body. Drag models are empirically developed as the specifics of how those collisions impact the moving body are quite complex. The two simplest models we have for drag are:

Linear Drag: $F_{lin} = \gamma v \rightarrow$ small & slow

Quadratic Drag: $F_{quad} = D v^2$ large & fast

Both point opposite the velocity, $-\hat{v} = \dfrac{-\vec{v}}{|v|}$

Critically: you need to consider your coordinate system

Example: Quadratic Drag (1D)

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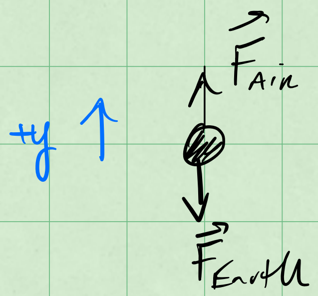

$$\vec{F}_{net} = \vec{F}_{AIR} + \vec{F}_{EARTH} = m\vec{a} = m\ddot{\vec{r}}$$

$$F_{Air} = Dv^2 \quad F_{EARTH} = mg$$

$$m\ddot{y} = -mg + Dv^2$$

EOM:

$$\boxed{\ddot{y} = -g + \frac{D}{m} v^2}$$

let $\tilde{D} = D/m$

How do we solve this EOM?

- separation of variables,
  we try to right each side as functions only of v and t

$$\int f(v)\,dv = \int g(t)\,dt$$

If we can integrate them we will get a closed form relationship between v and t.

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$$\ddot{y} = -g + \tilde{D}v^2 \implies \dot{v} = \frac{dv}{dt} = -g + \tilde{D}v^2$$

$$\frac{dv}{dt} = -g + \tilde{D}v^2$$

$$\frac{dv}{-g + \tilde{D}v^2} = dt$$

$$\frac{dv}{dt} = 0 = -g + \tilde{D}v^2 \implies v_t = \sqrt{g/\tilde{D}}$$

terminal velocity

$$\frac{dv}{g\left[(v/v_t)^2 - 1\right]} = dt$$

so $f(v) = \dfrac{1}{\left(\frac{v}{v_T}\right)^2 - 1}$

$g(t) = g$

$$\frac{dv}{\left(\frac{v}{v_t}\right)^2 - 1} = g\,dt$$

Drop from rest

$$\int_0^{v(t)} \frac{dv'}{\left(\frac{v'}{v_t}\right)^2 - 1} = \int_0^t g\,dt'$$

a trajectory requires initial conditions $v(0) = 0$

⑥

$$\int_0^{v} \frac{dv'}{\left(\frac{v'}{v_t}\right)^2 - 1} = \int_0^{t} g\,dt$$

→ look up integrals

⇒ WolframAlpha, sympy, etc.

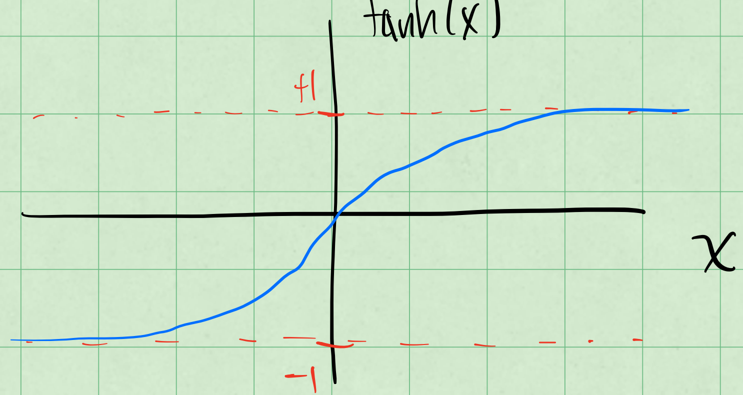
$$-v_t \tanh^{-1}\left(\frac{v}{v_t}\right) = gt$$

$$\boxed{v(t) = v_t \tanh\left(\frac{-gt}{v_t}\right)}$$

where $v_t = \sqrt{g/\tilde{D}} = \sqrt{mg/D}$

New equation! How do we check it?

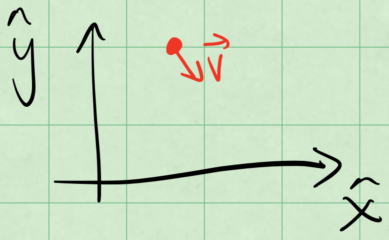
⇒ units? ⇒ limits?



Example: 2D Quadratic Drag

$$\vec{F}_D = -D\vec{v}|\vec{v}|$$

Need coordinate system.

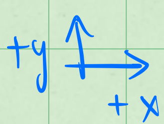


$$\vec{r} = x\hat{x} + y\hat{y} = \langle x, y \rangle$$

$$\vec{v} = V_y\hat{x} + V_y\hat{y} = \langle v_x, v_y \rangle$$

$$\vec{a} = a_x\hat{x} + a_y\hat{y} = \langle a_x, a_y \rangle$$

FBD



$$|\vec{v}| = \sqrt{V_x^2 + V_y^2}$$

Newton 2nd

$$ma_x = F_{AIR,x} \qquad ma_y = F_{AIR,y} - F_{EARTH}$$

$$F_{AIR,x} = -D|\vec{v}|v_x \qquad F_{EARTH} = +mg$$

$$F_{AIR,y} = -D|\vec{v}|v_y$$

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$$ma_x = m\ddot{x} = -Dv_x\sqrt{v_x^2+v_y^2}$$

$$ma_y = m\ddot{y} = -Dv_y\sqrt{v_x^2+v_y^2} - mg$$

Coupled EOMs (cannot go further)

$$\dot{v}_x = -\tilde{D}\,v_x\sqrt{v_x^2+v_y^2} \qquad \text{with } \tilde{D} = D/m$$

$$\dot{v}_y = -\tilde{D}\,v_y\sqrt{v_x^2+v_y^2} - g$$

We need another approach since there's no analytical solution to these Differential Equations

$\Rightarrow$ cannot form,

$$f_1(v_x)\,dv_x = g_1(t)\,dt$$

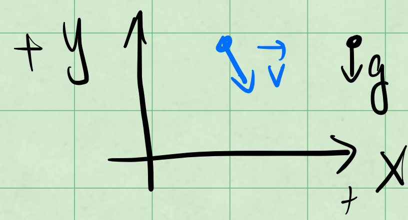
$$f_2(v_y)\,dv_y = g_2(t)\,dt$$

independent EOMs

so separation of variables is not possible

Example: Linear Drag in 2D

$$\vec{F}_{Lin} = -m\gamma\vec{v}$$



$$\vec{F}_{net} = \vec{F}_{lin} + \vec{F}_{EARTH}$$

$$ma_x = -m\gamma v_x$$

$$ma_y = -m\gamma v_y - mg$$

note v_x & v_y can be $+$, $-$, or, 0.
they are components

EOMs

$$\boxed{\dot{v}_x = -\gamma v_x \qquad \dot{v}_y = -\gamma v_y - g}$$

These are decoupled so we will try separation of variables

$$\dot{v}_x = \frac{dv_x}{dt} = -\gamma v_x$$

$$\frac{dv_x}{v_x} = -\gamma dt$$

Integrate!

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$$\int_{v_{0x}}^{v_x} \frac{dv_x'}{v_x'} = -\gamma \int_{t=0}^{t} dt'$$

$$\ln\left(v_x / v_{0x} \right) = -\gamma t$$

$$\boxed{v_x(t) = v_{0x} e^{-\gamma t}}$$

trajectory for v_x

$$\frac{dx}{dt} = v_x$$

$$\int_{x_0}^{x} dx' = \int_{t=0}^{t} v_{0x} e^{-\gamma t'} dt'$$

$$x - x_0 = -\frac{v_{0x}}{\gamma} \left(e^{-\gamma t'} \right)_0^t$$

$$x - x_0 = -\frac{v_{0x}}{\gamma} \left(e^{-\gamma t} - e^0 \right)$$

$$\boxed{x(t) = x_0 + \frac{1}{\gamma} v_{0x} \left(1 - e^{-\gamma t} \right)}$$

trajectory for x

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$$\dot{v}_y = -\gamma v_y - g$$

$$\frac{dv_y}{dt} = -\gamma v_y - g$$

$$\frac{dv_y}{\gamma v_y + g} = -dt \implies \frac{dv_y}{v_y + g/\gamma} = -\gamma dt$$

Integrate!!

$$\int \frac{dx}{a+x} = \ln(a+x) + C$$

$t=0$, $v_y = v_{0y}$

$$\int_{v_{0y}}^{v_y} \frac{dv_y'}{v_y' + g/\gamma} = \ln\left(\frac{g/\gamma + v_y}{g/\gamma + v_{0y}}\right) = -\gamma t$$

$$\left(\frac{g}{\gamma} + v_y\right) = \left(\frac{g}{\gamma} + v_{0y}\right)e^{-\gamma t}$$

$$\boxed{v_y(t) = \frac{g}{\gamma}\left(e^{-\gamma t} - 1\right) + v_{0y}e^{-\gamma t}}$$

trajectory for v_y.

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$$\frac{dy}{dt} = v_y$$

$$\int_{y_0}^{y} dy' = y - y_0 = \int_{t=0}^{t} \left\{ \frac{g}{\gamma}\left(e^{-\gamma t'} - 1\right) + v_{0y} e^{-\gamma t'} \right\} dt'$$

$$y - y_0 = \frac{g}{\gamma}\left(\frac{-1}{\gamma} e^{-\gamma t'} - t'\right)\Bigg|_0^t + v_{0y}\left(-\frac{1}{\gamma} e^{-\gamma t'}\right)\Bigg|_0^t$$

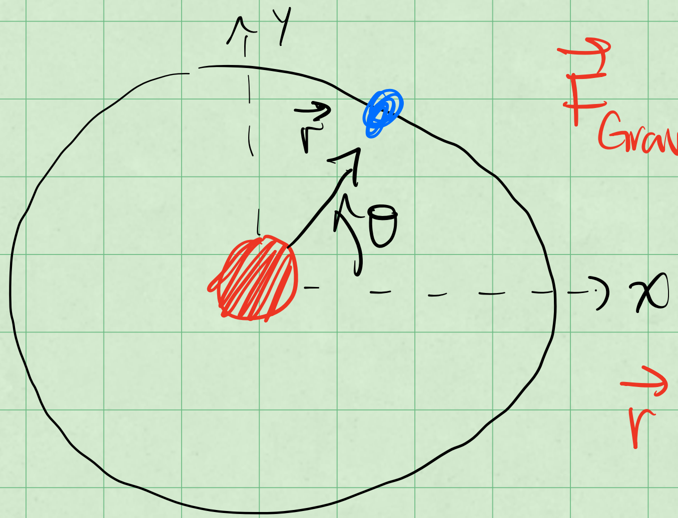
$$= \frac{g}{\gamma}\left(-\frac{1}{\gamma} e^{-\gamma t} - t + \frac{1}{\gamma}\right) + v_{0y}\left(-\frac{1}{\gamma} e^{-\gamma t} + \frac{1}{\gamma}\right)$$

$$y(t) = y_0 + \frac{v_{0y}}{\gamma} - \frac{v_{0y}}{\gamma} e^{-\gamma t} - \frac{g}{\gamma} t + \frac{g}{\gamma^2} - \frac{g}{\gamma^2} e^{-\gamma t}$$

$$\boxed{y(t) = y_0 - gt/\gamma + \frac{1}{\gamma}\left(v_{0y} + \frac{g}{\gamma}\right)\left(1 - e^{-\gamma t}\right)}$$

trajectory for y

Example: 2D Newtonian Gravitational Model (Sun - Earth)

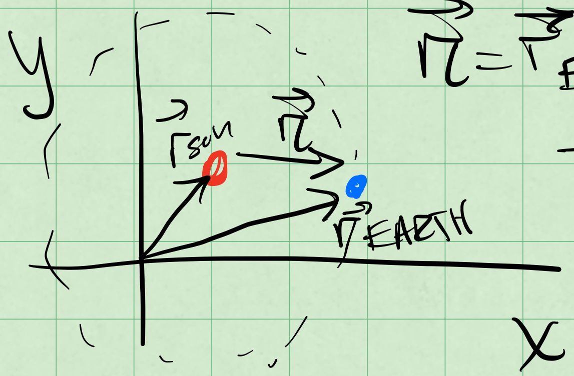


$$\vec{F}_{Grav} = \frac{-G m_1 m_2}{r^2}\hat{r}$$

$$\vec{r} = ?$$

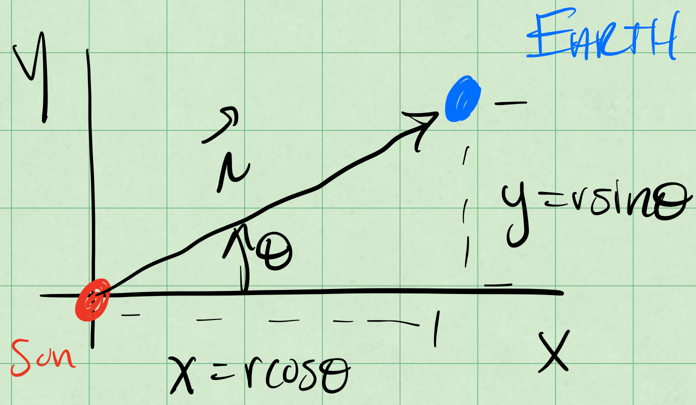
$$\vec{r} = \vec{r}_{object1} - \vec{r}_{object2}$$

Move Sun off $\langle 0, 0 \rangle$



$$\vec{r} = \vec{r}_{EARTH} - \vec{r}_{sun}$$

Back to the Simplified 2D Model



$M_\odot = 2 \cdot 10^{30}\,kg$

$M_E = 6 \cdot 10^{24}\,kg$

$G = 6.67 \cdot 10^{-11} \frac{N}{kg^2} m^3$

$$\vec{F}_{Grav} = -G \frac{M_\odot M_E}{|\vec{r}|^3} \vec{r}$$

$$|\vec{r}| = \sqrt{x^2+y^2} = 1\,A.U = 1.5 \cdot 10^{11}\,m$$

$$\vec{F}_{NEt} = \vec{F}_{Grav} = m\vec{a} = m\langle a_x\, a_y \rangle$$

$$F_x = \frac{-GM_\odot M_E x}{|r|^3} \qquad F_y = -\frac{GM_\odot M_E y}{|r|^3}$$

$$\boxed{\begin{aligned} a_x = \dot{v}_x = \dot{x} &= \frac{-GM_\odot x}{(x^2+y^2)^{3/2}} \\ a_y = \dot{v}_y = \dot{y} &= \frac{-GM_\odot y}{(x^2+y^2)^{3/2}} \end{aligned} \quad \text{EOMs for Grav (coupled)}}$$

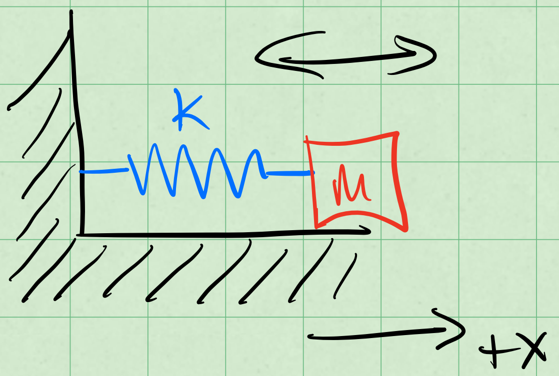
Trajectories?

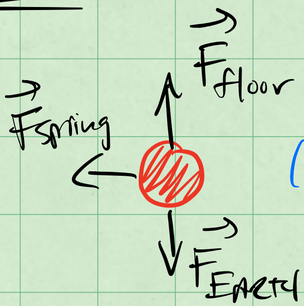
(1) Integrate EoMs

(2) solve coupled EoMs → techniques for this

(3) solve numerically

Example: Simple Harmonic Oscillator (1D)





(stretched right or compressed left)

$F_s = ks \leftarrow$ stretch $(x - L_0)$

relaxed length

if we measure displacement from equilibrium (L_0) then $F_s = kx$ direction opposite of stretch

$$F_x = ma_x = m\dot{v}_x = m\ddot{x} = -kx$$

$$\boxed{m\ddot{x} = -kx \quad \text{or} \quad \ddot{x} = -\frac{k}{m}x}$$

EOM of the SHO (Very important result)

Solutions?

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x \quad \text{where } \omega^2 = \frac{k}{m}$$

Try $x(t) = Ce^{i\omega t}$

$\dot{x}(t) = i\omega Ce^{i\omega t}$

$\ddot{x}(t) = -\omega^2 Ce^{i\omega t} = -\omega^2 x(t)$!!

$x(t) = Ce^{i\omega t}$ where $C = a + ib$ works!

So, maybe $\cos(\omega t)$ & $\sin(\omega t)$?

Try $x(t) = A\cos(\omega t) + B\sin(\omega t)$

$\dot{x}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$

$\ddot{x}(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

$= -\omega^2 x(t)$!!

$$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t) \text{ with } \omega^2 = k/m}$$

generic trajectory for SHO