Day 31 - Euler-Lagrange Equation

MICHIGAN STATE
UNIVERSITY

Welcome to PHY 321: Classical Mechanics

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Seminars this Week

WEDNESDAY, April 2, 2025

- **Astronomy Seminar**, 1:30 pm, 1400 BPS, *Andy Tzandikas, Univ. of Washington*, Title: Searching for the Rarest Stellar Occultations
- **PER Seminar**, 3:00 pm., BPS 1400, *Abigail Daane, Professor of Physics, South Seattle College*, Title: The obstacles, stumbles, and growth in examining the “decolonization” of physics education

THURSDAY, April 3, 2025

- **Colloquium**, 3:30 pm, 1415 BPS, *Alex Sushkov, Boston University*, Title: Nuclear magnetic resonance at the quantum sensitivity limit

Reminders

We proposed a solution to the line problem that involved an error term $\eta(x)$, which is a small perturbation to the true path $y(x)$. This leads to a perturbed function:

$$Y(x) = y(x) + \alpha\eta(x)$$

where α is a small parameter.

We proposed that there's a functional $f(Y, Y', x)$ that depends on a function $Y(x)$, its derivative $Y'(x)$, and the independent variable x such that:

$$\int_{s_1}^{s_2} f(Y, Y', x)\, dx > \int_{s_1}^{s_2} f(y, y', x)\, dx$$

Reminders

By taking the derivative of the functional with respect to α, we can find the condition for which the functional is stationary (i.e., a minimum or maximum).

$$\frac{d}{d\alpha}\int_{s_1}^{s_2} f(Y, Y', x)\, dx\bigg|_{\alpha=0} = 0$$

This (with a lot of math) led us to the following expression:

$$\int_{s_1}^{s_2} \eta(x)\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx = 0$$

Clicker Question 31-1

We completed this derivation with the following mathematical statement:

$$\int_{s_1}^{s_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] dx = 0$$

where $\eta(x)$ is an arbitrary function. What does this imply about the term in square brackets?

1. The term in square brackets must be a pure function of x.
2. The term in square brackets must be a pure function of y.
3. The term in square brackets must be a pure function of y'.
4. The term in square brackets must be zero.
5. The term in square brackets must be a non-zero constant.

Clicker Question 31-2

Returning to the line problem,

$$l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

here, $f(y, y', x) = \sqrt{1 + (y')^2}$, where $y' = \frac{dy}{dx}$.

Apply the Euler-Lagrange equation to find the expression for the function $f(y, y', x)$ in this case. Write your result to find the expression for the term in square brackets:

$$\frac{d}{dx}[?] = 0$$

Click when you have an answer!

Clicker Question 31-3

With,

$$y' = \pm\sqrt{\frac{c^2}{1+c^2}}$$

where c is a constant, the solution expresses a straight line.

1. True and I can prove it!
2. True, but I'm not sure how to prove it.
3. False, I think this is incorrect.
4. I don't know.

Clicker Question 31-4

We derived the time that it takes to run from a point on the shore to a point in the water, T:

$$T = \frac{1}{v_1}\left(x_1^2 + (y - y_1)^2\right)^{1/2} + \frac{1}{v_2}\left(x_2^2 + (y_2 - y)^2\right)^{1/2}$$

To find the minimal time, what derivative should we take?

1. $\frac{dT}{dx}$
2. $\frac{dT}{dy}$
3. $\frac{dT}{dt}$
4. Something else?

Clicker Question 31-5

For the brachistochrone problem, the ball moves purely under the influence of gravity. Consider that the ball has moved a vertical distance Δy from rest. What is the speed of the ball at this point?

1. $v = gt$
2. $v = 2g\Delta y$
3. $v = \sqrt{2g\Delta y}$
4. I'm not sure, but $< \sqrt{2g\Delta y}$
5. Something else?