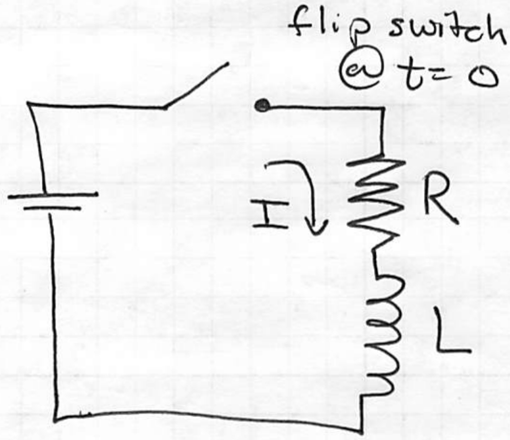
Example: An RL circuit.

The resistance might be distributed (in wires, battery, etc). And so might the inductance. This is a model.



This model will allow us to see the general solution method.

Intuitively, the inductor doesn't "like" instant changes in current. We expect that at $t=0$, the current slowly changes from zero. After a long time, there are no more changes. We've reached steady state! $\Delta V_{inductor} = 0$, so it acts like an ideal wire now.

Using Kirchoff's Loop Rule we find,

$$V - IR - L\frac{dI}{dt} = 0$$

Here, we assume that $V, R, \& L$ are all known and we are seeking $I(t)$.

This equation is a 1^{st} order, inhomogeneous ODE,

$$L\frac{dI}{dt} + IR = V$$

There are several methods to solve this ODE. We will discuss two

1. Direct method using homogeneous & particular solutions
2. Using the "phasors" method, which is very powerful and can be much simpler.

Method #1: Direct Solution (maybe remember from ODEs)

① Find the general homogeneous solution to:

$$L \frac{dI_H}{dt} + I_H R = 0$$

② Find some particular solution to the full (inhomog) eq'n.

③ Add these solutions ($I = I_H + I_p$) to get the full solution.

④ Determine the one arbitrary constant in I_H using initial conditions.

this method works for $V = V_0$ (battery)

also if $V = V_0 \cos(\omega t)$ (AC power supply)

and thus, by superposition, we can solve for any periodic $V(t)$ because Fourier says,

$$V(t) = \sum_n V_n \cos(\omega_n t + \delta_n)$$

* this method is fairly general.

Back to the example, the homogeneous equation is,

$$\frac{dI_H}{dt} = -\frac{R}{L} I_H \Rightarrow \text{seperates simply} \Rightarrow \frac{dI_H}{I_H} = -\frac{R}{L} dt$$

$$I_H(t) = I_H(t=0)\, e^{-Rt/L}$$

this is an undetermined constant.

the resistor, R, kills off the current while the inductor, L, stretches that time out.

To find particular solutions, you don't need generality. Any solution that works is the solution. Guess & check is just fine.

Let's say that $V = V_0 = \text{constant}$,

$$L \frac{dI_p}{dt} + I_p R = V_0$$

Given our homogeneous solution maybe something like this works,

$$I_p(t) = a e^{-Rt/L} + b$$ Let's check if this works.

$$\frac{dI_p}{dt} = -\frac{R}{L} a e^{-Rt/L}$$ so that,

$$L\left(-\frac{R}{L} a e^{-Rt/L}\right) + \left(a e^{-Rt/L} + b\right) R = V_0$$

the exponential terms cancel! this leaves

$bR = V_0$ so our proposed solution works if

$b = V_0/R$

Add the solutions together,

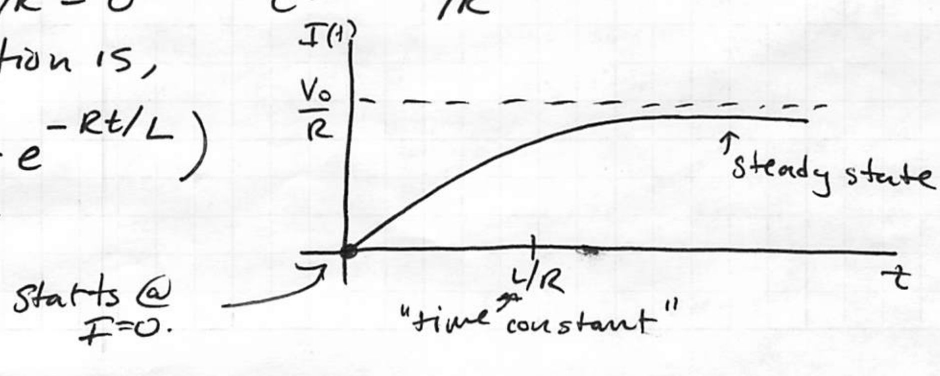
$$I(t) = I_p + I_H = \big(\underbrace{I_H(t=0) + a}_{\text{call this a constant, } c}\big) e^{-Rt/L} + V_0/R$$

$$I(t) = Ce^{-Rt/L} + V_0/R$$ if at $t=0$, $I=0$ then,

$$I(0) = C + V_0/R = 0 \qquad C = -V_0/R$$

so our solution is,

$$I(t) = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)$$



We will observe more interesting results when we have an AC supply. Let's work on this for,

$V(t) = V_0 \cos(\omega t)$

We have already found the homogeneous solution, I_H, so we just need a good guess for $I_p(t)$.

We'd expect that a sinusoidal source would result in a sinusoidal solution, so let's try,

$I_p(t) = a\cos(\omega t + \varphi)$ ← these are both undetermined coefficients.

Our differential equation is now,

$L\, dI_p/dt + I_p R = V_0 \cos(\omega t)$

$-La\omega \sin(\omega t + \varphi) + aR\cos(\omega t + \varphi) = V_0\cos(\omega t)$

this looks a little complex but we can simplify things with standard trig identities,

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

If we use these identities we find,

$$\begin{aligned} -La\omega \sin\omega t \cos\varphi - aR\sin\omega t \sin\varphi &= 0 \\ -La\omega \cos\omega t \sin\varphi + aR\cos\omega t\cos\varphi &= + \\ & \quad V_0\cos\omega t \end{aligned}$$

if the coeffs in front of the $\sin\omega t$ terms vanish and those in front of the $\cos\omega t$ terms give V_0, it works!

$$\left.\begin{aligned} -La\omega\cos\varphi - aR\sin\varphi &= 0 \\ -La\omega\sin\varphi + aR\cos\varphi &= V_0 \end{aligned}\right\} \text{Two eqn's and two unknowns } (a \,\&\, \varphi)$$

$\Rightarrow$ the first equation we have gives

$$-La\omega\cos\varphi = aR\sin\varphi$$

- if $a=0$, this works, but that means $I_p(t)=0$
- if instead, $\tan\varphi = \frac{-L\omega}{R}$ or $\boxed{\varphi = \tan^{-1}\left(-\frac{L\omega}{R}\right)}$

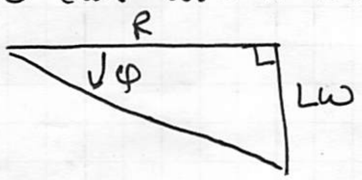
then we get a nonzero I_p.

So φ is not an arbitrary constant and is not dependent on initial conditions. It is determined by the circuit elements and the driver.

$\Rightarrow$ the second equation we have gives,

$$a(-L\omega\sin\varphi + R\cos\varphi) = V_0$$

We can use a triangle that shows $\tan\varphi = \frac{-L\omega}{R}$,



We can read off $\sin\varphi$ & $\cos\varphi$,

$$\sin\varphi = \frac{-L\omega}{\sqrt{R^2+L^2\omega^2}} \qquad \cos\varphi = \frac{+R}{\sqrt{R^2+L^2\omega^2}}$$

Let's put these back into the 2nd equation,

$$a\left(\frac{L^2\omega^2}{\sqrt{R^2+L^2\omega^2}} + \frac{R^2}{\sqrt{R^2+L^2\omega^2}}\right) = a\left(\sqrt{R^2+L^2\omega^2}\right) = V_0$$

$$\boxed{a = \frac{V_0}{\sqrt{R^2+L^2\omega^2}}}$$

So with $\varphi = \tan^{-1}\left(-\frac{L\omega}{R}\right)$ and $a = \frac{V_0}{\sqrt{R^2+L^2\omega^2}}$,

$I_p(t) = a\cos(\omega t + \varphi)$ works.

So our full solution is,

$$I(t) = I_p + I_H = a\cos(\omega t + \varphi) + I_{H0}\, e^{-Rt/L}$$

= persistent oscillatory response + Dying away piece

$\Rightarrow$ a and φ are determined already (on previous page)

I_{H0} is not determined; it is determined by initial conditions.

So if, for example, at $t=0$, $I=0$ then,

$$I(t>0) = a\cos(\omega t+\varphi) - \underbrace{a\cos\varphi}_{\text{makes } I(t=0)=0}\, e^{-Rt/L}$$

with amplitude $a = \dfrac{V_0}{\sqrt{R^2+L^2\omega^2}}$ and phase, $\varphi = \tan^{-1}\left(-\dfrac{L\omega}{R}\right)$

- when R is large, a is small. Big R kills off long term currents.
- when $\omega = 0$ (battery), $a \to V_0/R$ and $\tan^{-1}(0) = 0 = \varphi$.
  the inductor acts like an ideal wire in the long term limit w/ DC voltage.
- when $\omega \to \infty$, $a \to 0$; Inductors don't like rapid changes (big Back EMFs!)
  they will suppress response at high f.

This method works just fine, but it's a real pain when you have a more complex circuit. especially w/ multiple R's, L's, & C's in series and/or parallel.

Method 2 Phasors

- The phasor method is a bit more sophisticated, but it's incredibly powerful and widely used.
- It gets rid of the sines & cosines and changes our problem to a simple algebra problem using exponentials.
- We will make use of Euler's famous formula,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{or for our purposes,}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t.$$

What's nice about the exponential form is how they work under derivatives (and integrals),

$\frac{d}{dt}(\cos\omega t) = -\omega\sin\omega t$ is a new, linearly ind. function (leads to complications.)

But,

$\frac{d}{dt}(e^{i\omega t}) = i\omega e^{i\omega t}$, just proportional to the original function, $\frac{df}{dt} \propto f$. (much easier!)

So here's what we are going to do. Instead of $V_0\cos\omega t$ as the driver, we will use $V_0 e^{i\omega t}$. Now, this might bother you b/c the voltage is complex. That's fine, at the end of the day we will take the Real Part.

$$V_{true} = \mathrm{Re}\left[V_{fictious}\right] \quad \text{and} \quad I_{true} = \mathrm{Re}\left[I_{fictious}\right]$$

We can do this b/c the ODE is linear, so Re(I) arises from Re(V). The ODE will be simpler, but we will to remember to take the real part.

Lets rework the problem again using the phasor method,

$$L\frac{dI}{dt} + IR = V(t) = \tilde{V}e^{i\omega t}$$

so we have this (fictious) driving voltage, $\tilde{V}e^{i\omega t}$. It's complex

- the real voltage is $Re(\tilde{V}e^{i\omega t})$
- if $\tilde{V}$ is itself a complex constant (i.e., $\tilde{V} = V_0 e^{i\delta}$), then we can have more complex drivers $V_0\cos(\omega t+\delta)$

We know the solution for I_H, so we just need to find I_p. We will guess & check. this time we guess a simple form: $I_p = \tilde{I}e^{i\omega t}$

$$L\frac{dI_p}{dt} + I_p R = \tilde{V}e^{i\omega t} \quad \text{is the ODE,}$$

$$L\tilde{I}(i\omega)e^{i\omega t} + \tilde{I}Re^{i\omega t} = \tilde{V}e^{i\omega t} \quad \text{the } e^{i\omega t}\text{'s cancel out!}$$

$$L\tilde{I}(i\omega) + \tilde{I}R = \tilde{V} \quad \text{or} \quad \tilde{I} = \frac{\tilde{V}}{R+i\omega L}$$

that's it! $\tilde{I}$ is a constant and our solution is simply,

$$I_{true} = Re(I_{fictious}) = Re(\tilde{I}e^{i\omega t}) + I_H$$

from before

see how much simpler that is!

The solution looks like $\tilde{V} = \tilde{I}\tilde{R}$ with $\tilde{R}$ now complex

$\tilde{R}$ is the impedance (or complex impedance), we label it Z

for 

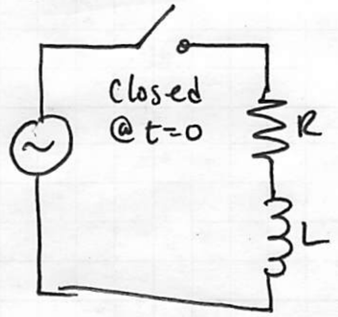
We got $Z = R + i\omega L$ } a series circuit just add the impedances } more general actually!

$Z_R = R$ resistor

$Z_L = i\omega L$ inductor

(turns out a capacitor has $Z_c = \frac{-i}{\omega C}$)

Let's return to the RL example and wrap it up,



$$Z_{tot} = R + i\omega L$$

so we have, $\tilde{V} = \tilde{I}(R + i\omega L)$

and $V_{true} = \mathrm{Re}\, \tilde{V} e^{i\omega t}$

$$I_{true} = \mathrm{Re}\, \tilde{I} e^{i\omega t} = \mathrm{Re}\left(\frac{\tilde{V} e^{i\omega t}}{R + i\omega L}\right)$$

In our original setup, $V_{real} = V_0 \cos \omega t$ so $\tilde{V} = V_0$

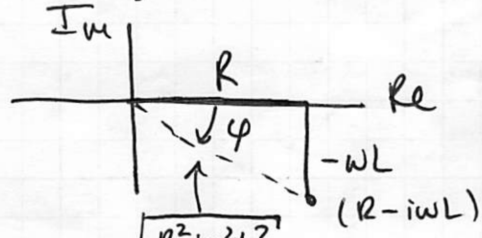
So we have:

$$I_{true} = \mathrm{Re}\left(\frac{V_0 e^{i\omega t}}{R + i\omega L}\right) = \mathrm{Re}\left(\frac{V_0 e^{i\omega t}}{R + i\omega L} \frac{R - i\omega L}{R - i\omega L}\right)$$

standard method

so, $$I = \frac{V_0}{R^2 + \omega^2 L^2} \mathrm{Re}\left(e^{i\omega t}(R - i\omega L)\right)$$

How do we deal with the rest of the expression? Another standard method, draw $R - i\omega L$ in the complex plane,



In the complex plane, this point is simply,

$$\sqrt{R^2 + \omega^2 L^2}\, e^{i\phi} \quad \text{with } \varphi = \tan^{-1}\left(\frac{-\omega L}{R}\right)$$ <u>as before</u>

so, $$I = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \mathrm{Re}\left(e^{i\omega t} e^{i\phi}\right) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \varphi)$$

this is the exact solution we had before but now $V = IZ$

this works for $V = V_0 \cos(\omega t + \delta) \Rightarrow$ use $\tilde{V} = V_0 e^{i\delta}$

or for $V = V_0 \sin(\omega t) \Rightarrow$ use $\tilde{V} = V_0 e^{i\pi/2}$